QUOTIENTS OF $\mathbb{C}^m - \{0\}$ BY DIAGONAL C*-ACTIONS

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Abstract. Let $q_1, \ldots, q_m$ be positive integers with $(q_1, \ldots, q_m) = 1$ and $\rho: \mathbb{C}^* \times \mathbb{C}^m \to \mathbb{C}^m$, $\rho(t, z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m)$ the diagonal C*-action on $\mathbb{C}^m$. Then the orbit space $\mathbb{C}^m - \{0\}/\mathbb{C}^*$ is a normal analytic space. In this paper, we shall show that $\mathbb{C}^m - \{0\}/\mathbb{C}^*$ has only rational singularities and, if $\delta(q_1, \ldots, q_m) < m - 3$ and $m > 3$, $\mathbb{C}^m - \{0\}/\mathbb{C}^*$ is rigid, where $\delta(q_1, \ldots, q_m)$ is the positive integer defined by $q_1, \ldots, q_m$.

1. Introduction. Let $M$ be a complex manifold and $G$ a complex Lie transformation group whose action is proper on $M$. Then the quotient space $M/G$ has only rational singularities and if codim $S(M/G) > 3$, $M/G$ is rigid, where $S(M/G)$ is the set of all singular points of $M/G$ [8, Theorem 1].

Let $q_1, \ldots, q_m$ be positive integers with $(q_1, \ldots, q_m) = 1$ and $\rho: \mathbb{C}^* \times \mathbb{C}^m \to \mathbb{C}^m$, $\rho(t, z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m)$ the diagonal C*-action on $\mathbb{C}^m$. Then the quotient space $\mathbb{C}^m - \{0\}/\mathbb{C}^*$ is a normal analytic space. In fact it is realized in complex projective space (G. Edmunds [2]). We put

$$q_i = (q_1, \ldots, q_i, \ldots, q_m), \quad q_i = q_i / q_i \cdots q_i$$

and

$$\delta(q_1, \ldots, q_m) = \text{Max}\{\delta; (q'_1, \ldots, q'_m) \neq 1\}.$$ 

In this paper, we shall show the following

Theorem. $\mathbb{C}^m - \{0\}/\mathbb{C}^*$ has only rational singularities. Moreover if $\delta(q_1, \ldots, q_m) < m - 3$ and $m > 3$, $\mathbb{C}^m - \{0\}/\mathbb{C}^*$ is rigid.

2. The proof of Theorem. We shall give some definitions. Let $(X, \theta_X)$ be an analytic space and $\pi: (\tilde{X}, \theta_{\tilde{X}}) \to (X, \theta_X)$ a resolution of singularities of $X$. Then $x \in X$ is called a rational singularity if $(R^n\pi_*\theta_{\tilde{X}})_x = 0$ for any $i > 0$ (cf. [1]). And $x \in X$ is called a rigid singularity if any local flat deformation of $(X, x)$ is locally trivial (cf. [7]).

Let $M$ be a complex manifold and $G$ a complex Lie transformation group on $M$. Then $G$ is called proper on $M$ if the graph mapping $\Psi: G \times M \to M \times M$, $\Psi(g, x) = (x, gx)$ is proper (cf. [3]).

Let $n, p_1, \ldots, p_m$ be positive integers with $(n, p_1, \ldots, p_m) = 1$, $n > 2$, and $G$ a finite cyclic group generated by $g$, acting on $\mathbb{C}^m$ as follows:

$$g(z_1, \ldots, z_m) = (e^{p_1}z_1, \ldots, e^{p_m}z_m).$$
where $e_n^{p_i} = \exp(2\pi \sqrt{-1} \ p_i/n)$ ($i = 1, \ldots, m$). Then the quotient space $\mathbb{C}^m/G$ is a normal analytic space and we call it the cyclic quotient of type $(n; p_1, \ldots, p_m)$.

$f \in \text{GL}(m, \mathbb{C})$ is called a reflection if $f$ is order finite and $\text{codim} \ F(f) = 1$, where $F(f) = \{ z \in \mathbb{C}^m; f(z) = z \}$.

We shall show the following

**Lemma.** The following three conditions are equivalent.

1. $\mathbb{C}^m/G$ is nonsingular.
2. $G$ is generated by reflections.
3. $n = \prod_{i=1}^m \tilde{p}_i$, where $\tilde{p}_i = (n, p_1, \ldots, \hat{p}_i, \ldots, p_m)$ ($i = 1, \ldots, m$).

**Proof.** (1) $\Leftrightarrow$ (2) is well known (cf. [5]).

(3) $\Rightarrow$ (2). From $n = \prod_{i=1}^m \tilde{p}_i > 2$, we may assume that $\tilde{p}_i, \ldots, \tilde{p}_k \neq 1$ and the others are equal to 1. Let

$$g_{ij} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} + e_{\tilde{p}_j}^{p_i} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} (j = 1, \ldots, k)$$

be the diagonal matrix with $e_{\tilde{p}_j}^{p_i}$ as $(i, j)$-element and $G_j$ be the group generated by the reflection $g_{ij}$. Then

$$g_{ij} = g^{n/\tilde{p}_j}$$

and so $G_j$ is a subgroup of $G$ ($j = 1, \ldots, k$). We have $\text{ord}(G) = n$ from $(n, p_1, \ldots, p_m) = 1$ and $\text{ord}(G_j) = \tilde{p}_j$ from $(\tilde{p}_j, p_j) = 1$. Thus

$$\text{ord}(G) = \prod_{j=1}^k \text{ord}(G_j).$$

Since $(\tilde{p}_j, \tilde{p}_j) = 1 (j \neq j')$, $G$ must be generated by $g_{i_1}, \ldots, g_{i_k}$. So (3) implies (2).

(2) $\Rightarrow$ (3). If all $\tilde{p}_i$ are equal to 1, $G$ has no reflection. Thus we may also assume that $\tilde{p}_i, \ldots, \tilde{p}_k \neq 1$ and the others are equal to 1. The above argument shows that $G$ contains the subgroup generated by reflections $g_{i_1}, \ldots, g_{i_k}$. We show that $G$ has no other reflection. If there exist $i, r$ such that

$$g^r = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

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is a reflection, it is enough to show that \( r \) is a multiple of \( n/\tilde{p}_1 \). Since 

\[ p_1r, \ldots, p_{i-1}r, p_{i+1}r, \ldots, p_mr \]

are multiples of \( n \), \((p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m)r \) is a multiple of \( n \). Hence \( \tilde{p}_1r \) is a multiple of \( n \) and so \( r \) is a multiple of \( n/\tilde{p}_1 \). Therefore \( G \) is generated by \( g_{i_1}, \ldots, g_{i_k} \). From \((\tilde{p}_1, \tilde{p}_2) = 1 \) \((j \neq j')\), we have \( \text{ord}(G) = \prod_{j=1}^{m} \text{ord}(G_j) \). Thus \( n = \prod_{j=1}^{m} \tilde{p}_j \). Q.E.D.

Remark. The topological statement between (1) and (3) is proved by W. Neumann [4].

Now we can prove Theorem by the technique of R. Randell [6]. Since \( q_1, \ldots, q_m \) are positive, the \( C^* \)-action \( \rho: C^* \times C^m \to C^m, \rho(t, z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m) \) is proper on \( C^m \). Therefore \( C^m - \{0\}/C^* \) has only rational singularities by Theorem 1 [8].

For any \( x = (x_1, \ldots, x_m) \in C^m - \{0\} \), we may assume that \( x_1, \ldots, x_\delta \neq 0, x_{\delta+1} = \ldots = x_m = 0 \). The isotropy group \( (C^*)_x = \{ t \in C^*; x = \rho(t, x) \} \) is the cyclic group \( \mathbb{Z}_n = \{ t; t^n = 1 \} \), where \( n = (q_1, \ldots, q_\delta) \). Since the \( C^* \)-action is proper on \( C^m \), by H. Holmann [3] there exist an open connected neighborhood \( U \) of \( x \) and a submanifold \( N \subset U \) such that

(1) \( x \in N \),

(2) \( U/C^* = N/(C^*)_x \).

Here we may take \( N = \{ z_1 = x_1 \} \) and \( (C^*)_x \) acts on \( N \) as follows:

\[ \rho': (C^*)_x \times N \to N, \]

\[ \rho'(t, z_2, \ldots, z_\delta, z_{\delta+1}, \ldots, z_m) = (z_2, \ldots, z_\delta, t^{q_{\delta+1}}z_{\delta+1}, \ldots, t^{q_m}z_m). \]

Thus \( N/(C^*)_x \) is a product of \( C^{\delta-1} \) and the cyclic quotient of type \((n; q_{\delta+1}, \ldots, q_m)\). By Lemma, \( N/(C^*)_x \) is nonsingular if and only if \( n = q_{\delta+1} \cdots q_m \). From \((q_1, \ldots, q_\delta) = q_{\delta+1} \cdots q_m(q'_1, \ldots, q'_\delta), n = q_{\delta+1} \cdots q_m = 1 \) is equivalent to \((q'_1, \ldots, q'_\delta) = 1 \). We have

\[ \dim S(C^m - \{0\}/C^*) = \max\{-1, \delta(q_1, \ldots, q_m) - 1\} \]

by the definition of \( \delta(q_1, \ldots, q_m) \) (cf. [6]). Therefore if \( \delta(q_1, \ldots, q_m) < m - 3 \) and \( m > 3 \), \( C^m - \{0\}/C^* \) is nonsingular or codim \( S(C^m - \{0\}/C^*) \geq 3 \). Thus \( C^m - \{0\}/C^* \) has only rigid singularities by Theorem 1 [8]. Q.E.D.

Remark. For \( m = 1, 2, C^m - \{0\}/C^* \) is nonsingular. And for \( m > 3 \), \( C^m - \{0\}/C^* \) is not always rigid if \( \delta(q_1, \ldots, q_m) < m - 3 \). Let \( q_1 = 2, q_2 = q_3 = 3 \). Then \( q_1 = 3, q_2 = 3, q_3 = 3 \) and \( q_1' = 2, q_2' = q_3' = 1 \). So we have \( \delta(q_1, q_2, q_3) = 1 \).

Let \( x = (x_1, 0, 0), x_1 \neq 0 \). Then \( (C^*)_x = \{ t; t^2 = 1 \} \) and \( N = \{ z_1 = x_1 \} \). Hence \( N/(C^*)_x \) is the cyclic quotient of type \((2, 3, 3) \approx (2, 1, 1) \) and \( N/(C^*)_x \approx \{ v^2 - uw = 0 \} \subset C^3 \). Therefore \( C^3 - \{0\}/C^* \) is not rigid.

References


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