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# QUOTIENTS OF $\mathbf{C}^m - \{0\}$ BY DIAGONAL $\mathbf{C}^*$-ACTIONS

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ABSTRACT. Let $q_1, \ldots, q_m$ be positive integers with $(q_1, \ldots, q_m) = 1$ and $\rho$: $\mathbf{C}^* \times \mathbf{C}^m \to \mathbf{C}^m$, $\rho(t, z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m)$ the diagonal $\mathbf{C}^*$-action on $\mathbf{C}^m$. Then the orbit space $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is a normal analytic space. In this paper, we shall show that $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ has only rational singularities and, if $\delta(q_1, \ldots, q_m) < m - 3$ and $m > 3$, $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is rigid, where $\delta(q_1, \ldots, q_m)$ is the positive integer defined by $q_1, \ldots, q_m$.

**1. Introduction.** Let $M$ be a complex manifold and $G$ a complex Lie transformation group whose action is proper on $M$. Then the quotient space $M/G$ has only rational singularities and if codim $S(M/G) > 3$, $M/G$ is rigid, where $S(M/G)$ is the set of all singular points of $M/G$ [8, Theorem 1].

Let $q_1, \ldots, q_m$ be positive integers with $(q_1, \ldots, q_m) = 1$ and $\rho$: $\mathbf{C}^* \times \mathbf{C}^m \to \mathbf{C}^m$, $\rho(t, z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m)$ the diagonal $\mathbf{C}^*$-action on $\mathbf{C}^m$. Then the quotient space $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is a normal analytic space. In fact it is realized in complex projective space (G. Edmunds [2]). We put

$$\bar{q}_i = (q_1, \ldots, \hat{q}_i, \ldots, q_m), \qquad q_i' = q_i/\bar{q}_1 \cdots \hat{\bar{q}}_i \cdots \bar{q}_m$$

and

$$\delta(q_1, \ldots, q_m) = \operatorname{Max}\{\delta;\ (q'_{i_1}, \ldots, q'_{i_\delta}) \neq 1\}.$$

In this paper, we shall show the following

THEOREM. *$\mathbf{C}^m - \{0\}/\mathbf{C}^*$ has only rational singularities. Moreover if $\delta(q_1, \ldots, q_m) < m - 3$ and $m > 3$, $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is rigid.*

**2. The proof of Theorem.** We shall give some definitions. Let $(X, \mathcal{O}_X)$ be an analytic space and $\pi$: $(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to (X, \mathcal{O}_X)$ a resolution of singularities of $X$. Then $x \in X$ is called a rational singularity if $(R^i\pi_*\mathcal{O}_{\tilde{X}})_x = 0$ for any $i > 0$ (cf. [1]). And $x \in X$ is called a rigid singularity if any local flat deformation of $(X, x)$ is locally trivial (cf. [7]).

Let $M$ be a complex manifold and $G$ a complex Lie transformation group on $M$. Then $G$ is called proper on $M$ if the graph mapping $\Psi$: $G \times M \to M \times M$, $\Psi(g, x) = (x, gx)$ is proper (cf. [3]).

Let $n, p_1, \ldots, p_m$ be positive integers with $(n, p_1, \ldots, p_m) = 1$, $n > 2$, and $G$ a finite cyclic group generated by $g$, acting on $\mathbf{C}^m$ as follows:

$$g(z_1, \ldots, z_m) = (e_n^{p_1}z_1, \ldots, e_n^{p_m}z_m),$$

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where $e_n^{p_i} = \exp(2\pi\sqrt{-1}\, p_i/n)$ $(i = 1, \ldots, m)$. Then the quotient space $\mathbf{C}^m/G$ is a normal analytic space and we call it the cyclic quotient of type $(n; p_1, \ldots, p_m)$.

$f \in GL(m, \mathbf{C})$ is called a reflection if $f$ is order finite and codim $F(f) = 1$, where $F(f) = \{z \in \mathbf{C}^m; f(z) = z\}$.

We shall show the following

LEMMA. *The following three conditions are equivalent.*

(1) $\mathbf{C}^m/G$ *is nonsingular.*

(2) $G$ *is generated by reflections.*

(3) $n = \prod_{i=1}^m \bar{p}_i$, *where* $\bar{p}_i = (n, p_1, \ldots, \hat{p}_i, \ldots, p_m)$ $(i = 1, \ldots, m)$.

PROOF. (1) $\Leftrightarrow$ (2) is well known (cf. [5]).

(3) $\Rightarrow$ (2). From $n = \prod_{i=1}^m \bar{p}_i \geq 2$, we may assume that $\bar{p}_{i_1}, \ldots, \bar{p}_{i_k} \neq 1$ and the others are equal to 1. Let

$$g_{i_j} = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & e_{\bar{p}_{i_j}}^{p_{i_j}} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \qquad (j = 1, \ldots, k)$$

be the diagonal matrix with $e_{\bar{p}_{i_j}}^{p_{i_j}}$ as $(i_j, i_j)$-element and $G_{i_j}$ be the group generated by the reflection $g_{i_j}$. Then

$$g_{i_j} = g^{n/\bar{p}_{i_j}}$$

and so $G_{i_j}$ is a subgroup of $G$ $(j = 1, \ldots, k)$. We have $\mathrm{ord}(G) = n$ from $(n, p_1, \ldots, p_m) = 1$ and $\mathrm{ord}(G_{i_j}) = \bar{p}_{i_j}$ from $(\bar{p}_{i_j}, p_{i_j}) = 1$. Thus

$$\mathrm{ord}(G) = \prod_{j=1}^{k} \mathrm{ord}(G_{i_j}).$$

Since $(\bar{p}_{i_j}, \bar{p}_{i_{j'}}) = 1$ $(j \neq j')$, $G$ must be generated by $g_{i_1}, \ldots, g_{i_k}$. So (3) implies (2).

(2) $\Rightarrow$ (3). If all $\bar{p}_i$ are equal to 1, $G$ has no reflection. Thus we may also assume that $\bar{p}_{i_1}, \ldots, \bar{p}_{i_k} \neq 1$ and the others are equal to 1. The above argument shows that $G$ contains the subgroup generated by reflections $g_{i_1}, \ldots, g_{i_k}$. We show that $G$ has no other reflection. If there exist $i, r$ such that

$$g^r = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & e_n^{p_i r} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$$

is a reflection, it is enough to show that $r$ is a multiple of $n/\bar{p}_i$. Since $p_1 r, \ldots, p_{i-1}r, p_{i+1}r, \ldots, p_m r$ are multiples of $n$, $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m)r$ is a multiple of $n$. Hence $\bar{p}_i r$ is a multiple of $n$ and so $r$ is a multiple of $n/\bar{p}_i$. Therefore $G$ is generated by $g_{i_1}, \ldots, g_{i_k}$. From $(\bar{p}_{i_j}, \bar{p}_{i_{j'}}) = 1$ $(j \neq j')$, we have $\operatorname{ord}(G) = \prod_{j=1}^k \operatorname{ord}(G_{i_j})$. Thus $n = \prod_{i=1}^m \bar{p}_i$. Q.E.D.

REMARK. The topological statement between (1) and (3) is proved by W. Neumann [4].

Now we can prove Theorem by the technique of R. Randell [6]. Since $q_1, \ldots, q_m$ are positive, the $\mathbf{C}^*$-action $\rho$: $\mathbf{C}^* \times \mathbf{C}^m \to \mathbf{C}^m$, $\rho(t, z_1, \ldots, z_m) = (t^{q_1}z_1, \ldots, t^{q_m}z_m)$ is proper on $\mathbf{C}^m$. Therefore $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ has only rational singularities by Theorem 1 [8].

For any $x = (x_1, \ldots, x_m) \in \mathbf{C}^m - \{0\}$, we may assume that $x_1, \ldots, x_\delta \neq 0$, $x_{\delta+1} = \ldots = x_m = 0$. The isotropy group $(\mathbf{C}^*)_x = \{t \in \mathbf{C}^*; x = \rho(t, x)\}$ is the cyclic group $\mathbf{Z}_n = \{t; t^n = 1\}$, where $n = (q_1, \ldots, q_\delta)$. Since the $\mathbf{C}^*$-action is proper on $\mathbf{C}^m$, by H. Holmann [3] there exist an open connected neighborhood $U$ of $x$ and a submanifold $N \subset U$ such that

(1) $x \in N$,

(2) $U/\mathbf{C}^* \cong N/(\mathbf{C}^*)_x$.

Here we may take $N = \{z_1 = x_1\}$ and $(\mathbf{C}^*)_x$ acts on $N$ as follows:

$$\rho' \colon (\mathbf{C}^*)_x \times N \to N,$$

$$\rho'(t, z_2, \ldots, z_\delta, z_{\delta+1}, \ldots, z_m) = (z_2, \ldots, z_\delta, t^{q_{\delta+1}}z_{\delta+1}, \ldots, t^{q_m}z_m).$$

Thus $N/(\mathbf{C}^*)_x$ is a product of $\mathbf{C}^{\delta-1}$ and the cyclic quotient of type $(n; q_{\delta+1}, \ldots, q_m)$. By Lemma, $N/(\mathbf{C}^*)_x$ is nonsingular if and only if $n = \bar{q}_{\delta+1} \cdots \bar{q}_m$. From $(q_1, \ldots, q_\delta) = \bar{q}_{\delta+1} \cdots \bar{q}_m(q_1', \ldots, q_\delta')$, $n = \bar{q}_{\delta+1} \cdots \bar{q}_m$ is equivalent to $(q_1', \ldots, q_\delta') = 1$. We have

$$\dim S(\mathbf{C}^m - \{0\}/\mathbf{C}^*) = \operatorname{Max}\{-1, \delta(q_1, \ldots, q_m) - 1\}$$

by the definition of $\delta(q_1, \ldots, q_m)$ (cf. [6]). Therefore if $\delta(q_1, \ldots, q_m) < m - 3$ and $m > 3$, $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is nonsingular or codim $S(\mathbf{C}^m - \{0\}/\mathbf{C}^*) > 3$. Thus $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ has only rigid singularities by Theorem 1 [8]. Q.E.D.

REMARK. For $m = 1, 2$, $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is nonsingular. And for $m > 3$, $\mathbf{C}^m - \{0\}/\mathbf{C}^*$ is not always rigid if $\delta(q_1, \ldots, q_m) \nless m - 3$. Let $q_1 = 2, q_2 = q_3 = 3$. Then $\bar{q}_1 = 3, \bar{q}_2 = \bar{q}_3 = 1$ and $q_1' = 2, q_2' = q_3' = 1$. So we have $\delta(q_1, q_2, q_3) = 1$. Let $x = (x_1, 0, 0)$, $x_1 \neq 0$. Then $(\mathbf{C}^*)_x = \{t; t^2 = 1\}$ and $N = \{z_1 = x_1\}$. Hence $N/(\mathbf{C}^*)_x$ is the cyclic quotient of type $(2; 3, 3) \cong (2; 1, 1)$ and $N/(\mathbf{C}^*)_x \cong \{v^2 - uw = 0\} \subset \mathbf{C}^3$. Therefore $\mathbf{C}^3 - \{0\}/\mathbf{C}^*$ is not rigid.

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