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A REMARK ON THE F. AND M. RIESZ THEOREM

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ABSTRACT. Let μ be a measure of analytic type on the unit circle. We give a short direct proof that μ^2 is absolutely continuous with respect to Lebesgue measure; our method also gives a convolution product version of some related several variable results.

Let $\mathbf{T}$ be the circle group, $\mathbf{Z}$ the integers and $M(\mathbf{T})$ the complex-valued regular Borel measures on $\mathbf{T}$. For $\mu \in M(\mathbf{T})$ and $n \in \mathbf{Z}$ define $\hat{\mu}(n) = \int_{\mathbf{T}} e^{-in\theta}\, d\mu(\theta)$. Denote by $M_a(\mathbf{T})$ those $\mu \in M(\mathbf{T})$ which are absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$. We now cite the theorem of F. and M. Riesz.

THEOREM 1. *Let $\mu \in M(\mathbf{T})$ such that $\hat{\mu}(n) = 0$ for all $n < 0$. Then $\mu \in M_a(\mathbf{T})$.*

Several short and elegant proofs of Theorem 1 are known; see for example [1] and [2]. In this note, we prove using only the most elementary facts of harmonic analysis that the conclusion of Theorem 1 holds for convolution products. More precisely:

THEOREM 2. *Let $\mu \in M(\mathbf{T})$ such that $\hat{\mu}(n) = 0$ for all $n < 0$. Then $\mu^2 \in M_a(\mathbf{T})$.*

PROOF. For $\nu \in M(\mathbf{T})$ and $E \subset \mathbf{Z}$ put $\|\nu\|_{B(E)} = \inf_\rho\{\|\rho\|: \rho \in M(\mathbf{T}), \hat{\rho} = \hat{\nu}$ on $E\}$. Then if $\mu \in M(\mathbf{T})$ with $\hat{\mu}(n) = 0$ for all $n < 0$ it follows that for all integers $N > 0$,

$$\|\mu\|_{B(-\infty,-N]} = 0. \tag{1}$$

Let $d\tau = t\,d\mu$ where t is a trigonometric polynomial on $\mathbf{T}$. An easy consequence of (1) is that $\lim_{N\to\infty}\|\tau\|_{B(-\infty,-N]} = 0$; this implies that if $\omega \in M(\mathbf{T})$ and $\omega \ll \mu$ then $\lim_{N\to\infty}\|\omega\|_{B(-\infty,-N]} = 0$. Let $\nu = |\mu|$; since $\nu \ll \mu$ we obtain $\lim_{N\to\infty}\|\nu\|_{B(-\infty,-N]} = 0$ and inasmuch as ν is a real measure we see that $\lim_{N\to\infty}\|\nu\|_{B[N,+\infty)} = 0$. Now $\lim_{N\to\infty}\|\nu\|_{B[N,+\infty)} = 0$ implies that

$$\lim_{N\to\infty}\|\mu\|_{B[N,+\infty)} = 0. \tag{2}$$

Let $\varepsilon > 0$ be given. We gather from (2) that there is a measure μ_ε and a positive integer $N(\varepsilon)$ satisfying

$$\|\mu_\varepsilon\| < \varepsilon \quad \text{and} \quad \hat{\mu}(n) = \hat{\mu}_\varepsilon(n) \text{ for all } n > N(\varepsilon). \tag{3}$$

Notice that because of (1) and the second part of (3), $\mu * (\mu - \mu_\varepsilon)$ is a trigonometric polynomial. Inasmuch as $M_a(\mathbf{T})$ is closed, we conclude from (3) that

$$\lim_{\varepsilon\to 0}\{\mu * (\mu - \mu_\varepsilon)\} = \mu^2 \in M_a(\mathbf{T}).$$

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COMMENTS. (a) It is trivial that Theorem 1 implies Theorem 2. Let $H^1(\mathbf{T})$ denote the measures of analytic type. A set $S \subset \mathbf{Z}$ is called a small p set if whenever $\mu \in M(\mathbf{T})$ with supp $\hat{\mu} \subset S \Rightarrow \mu^p \in M_a(\mathbf{T})$; small 1 sets are called Riesz sets. Theorem 2 would imply Theorem 1 if it could be directly proved that $H^1(\mathbf{T}) = H^1(\mathbf{T}) * H^1(\mathbf{T})$ or that every small 2 set is a Riesz set.

(b) The proof of Theorem 2 can be easily adapted to give the following three results:

(i) Let $\mathbf{R}$ be the real line and suppose $\mu \in M(\mathbf{R})$ such that $\hat{\mu}(x) = 0$ for all $x < 0$. Then $\mu^2 \in M_a(\mathbf{R})$.

(ii) Let S be a plane sector of angular opening less than π radians. If $\mu \in M(\mathbf{T}^2)$ and supp $\hat{\mu} \subset S$ then $\mu^2 \in M_a(\mathbf{T}^2)$.

(iii) For n a natural number put $Q_n = \{(m_1, \ldots, m_n)\colon m_i > 0, m_i \in \mathbf{Z}\}$. If $\mu \in M(\mathbf{T}^n)$ and supp $\hat{\mu} \subset Q_n$ then $\mu^2 \in M_a(\mathbf{T}^n)$.

(c) The argument in Theorem 2 that formula (1) implies formula (2) is due essentially to A. Rajchman.

REFERENCES

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