A REMARK ON THE F. AND M. RIESZ THEOREM
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Abstract. Let $\mu$ be a measure of analytic type on the unit circle. We give a short direct proof that $\mu^2$ is absolutely continuous with respect to Lebesgue measure; our method also gives a convolution product version of some related several variable results.

Let $T$ be the circle group, $\mathbb{Z}$ the integers and $M(T)$ the complex-valued regular Borel measures on $T$. For $\mu \in M(T)$ and $n \in \mathbb{Z}$ define $\hat{\mu}(n) = f_n e^{-int} d\mu(\theta)$. Denote by $M_a(T)$ those $\mu \in M(T)$ which are absolutely continuous with respect to Lebesgue measure on $T$. We now cite the theorem of F. and M. Riesz.

Theorem 1. Let $\mu \in M(T)$ such that $\hat{\mu}(n) = 0$ for all $n < 0$. Then $\mu \in M_a(T)$.

Several short and elegant proofs of Theorem 1 are known; see for example [1] and [2]. In this note, we prove using only the most elementary facts of harmonic analysis that the conclusion of Theorem 1 holds for convolution products. More precisely:

Theorem 2. Let $\mu \in M(T)$ such that $\hat{\mu}(n) = 0$ for all $n < 0$. Then $\mu^2 \in M_a(T)$.

Proof. For $\nu \in M(T)$ and $E \subset \mathbb{Z}$ put $\|\nu\|_{B(E)} = \inf\{\|\rho\|: \rho \in M(T), \hat{\rho} = \hat{\nu} \text{ on } E\}$. Then if $\mu \in M(T)$ with $\hat{\mu}(n) = 0$ for all $n < 0$ it follows that for all integers $N > 0$,

$$\|\mu\|_{B(-\infty, -N)} = 0. \quad (1)$$

Let $d\tau = t d\mu$ where $t$ is a trigonometric polynomial on $T$. An easy consequence of (1) is that $\lim_{N \to \infty} \|\tau\|_{B(-\infty, -N)} = 0$; this implies that if $\omega \in M(T)$ and $\omega \ll \mu$ then $\lim_{N \to \infty} \|\omega\|_{B(-\infty, -N)} = 0$. Let $\nu = |\mu|$; since $\nu \ll \mu$ we obtain $\lim_{N \to \infty} \|\nu\|_{B(-\infty, -N)} = 0$ and inasmuch as $\nu$ is a real measure we see that $\lim_{N \to \infty} \|\nu\|_{B(N, +\infty)} = 0$. Now $\lim_{N \to \infty} \|\nu\|_{B(N, +\infty)} = 0$ implies that

$$\lim_{N \to \infty} \|\mu\|_{B(N, +\infty)} = 0. \quad (2)$$

Let $\varepsilon > 0$ be given. We gather from (2) that there is a measure $\mu_\varepsilon$ and a positive integer $N(\varepsilon)$ satisfying

$$\|\mu_\varepsilon\| < \varepsilon \quad \text{and} \quad \hat{\mu}(n) = \hat{\mu}_\varepsilon(n) \text{ for all } n \geq N(\varepsilon). \quad (3)$$

Notice that because of (1) and the second part of (3), $\mu \ast (\mu - \mu_\varepsilon)$ is a trigonometric polynomial. Inasmuch as $M_a(T)$ is closed, we conclude from (3) that

$$\lim_{\varepsilon \to 0} \{\mu \ast (\mu - \mu_\varepsilon)\} = \mu^2 \in M_a(T).$$

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COMMENTS. (a) It is trivial that Theorem 1 implies Theorem 2. Let $H^1(T)$ denote the measures of analytic type. A set $S \subset \mathbb{Z}$ is called a small $p$ set if whenever $\mu \in M(T)$ with $\text{supp} \hat{\mu} \subset S \Rightarrow \mu^p \in M_a(T)$; small 1 sets are called Riesz sets. Theorem 2 would imply Theorem 1 if it could be directly proved that $H^1(T) = H^1(T) \ast H^1(T)$ or that every small 2 set is a Riesz set.

(b) The proof of Theorem 2 can be easily adapted to give the following three results:

(i) Let $\mathbb{R}$ be the real line and suppose $\mu \in M(\mathbb{R})$ such that $\hat{\mu}(x) = 0$ for all $x < 0$. Then $\mu^2 \in M_a(\mathbb{R})$.

(ii) Let $S$ be a plane sector of angular opening less than $\pi$ radians. If $\mu \in M(T^2)$ and $\text{supp} \hat{\mu} \subset S$ then $\mu^2 \in M_a(T^2)$.

(iii) For $n$ a natural number put $Q_n = \{(m_1, \ldots, m_n): m_i > 0, m_i \in \mathbb{Z}\}$. If $\mu \in M(T^n)$ and $\text{supp} \hat{\mu} \subset Q_n$ then $\mu^2 \in M_a(T^n)$.

(c) The argument in Theorem 2 that formula (1) implies formula (2) is due essentially to A. Rajchman.

REFERENCES


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