PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, August 1980

ON A CERTAIN C^*-CROSSED PRODUCT INSIDE A W^*-CROSSED PRODUCT

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ABSTRACT. To each W^*-dynamical system $(\mathfrak{M}, G, \alpha)$ corresponds canonically a C^*-dynamical system $(\mathfrak{M}^c, G, \alpha|\mathfrak{M}^c)$. We show that the C^*-crossed product $G \times_\alpha \mathfrak{M}^c$ can be identified with a certain C^*-subalgebra of the W^*-crossed product $G \times_\alpha \mathfrak{M}$.

The major part of the theory of noncommutative dynamical systems and their crossed products is Takesaki's work; see e.g. [7] and [8]. An important contribution, however, was made by Landstad who in [1] characterized those operator algebras that are crossed products with a given locally compact group G. Landstad's theory of G-products for abelian groups was exploited in [4] and [2] and we shall use it again to solve a problem arising from the difference between C^*- and W^*-crossed products.

A general exposition of noncommutative dynamical systems can be found in Chapters 7 and 8 of [5], but only the elementary parts of the theory will be needed here. Recall that a triple $(\mathcal{A}, G, \alpha)$ is called a C^*-dynamical system if $\mathcal{A}$ is a C^*-algebra and α is a representation of the locally compact abelian group G as automorphisms on $\mathcal{A}$, such that each function $t \to \alpha_t(x)$, $x \in \mathcal{A}$, is norm continuous. If $\mathfrak{M}$ is a von Neumann algebra we define analogously a W^*-dynamical system $(\mathfrak{M}, G, \alpha)$, but now only with the requirement that each function $t \to \alpha_t(x)$, $x \in \mathfrak{M}$, is σ-weakly continuous.

Given a W^*-dynamical system $(\mathfrak{M}, G, \alpha)$ define $\mathfrak{M}^c$ to be the set of elements x in $\mathfrak{M}$ for which the function $t \to \alpha_t(x)$ is norm continuous, see [5, 7.5.1]. Clearly $\mathfrak{M}^c$ is a G-invariant C^*-subalgebra of $\mathfrak{M}$ containing all elements of the form

$$\alpha_f(y) = \int \alpha_t(y) f(t)\, dt, \qquad y \in \mathfrak{M}, f \in L^1(G)$$

(since translation is continuous on $L^1(G)$). Using an approximate unit in $L^1(G)$ we see that $\mathfrak{M}^c$ is in fact generated by elements $\alpha_f(y)$, and therefore σ-weakly dense in $\mathfrak{M}$. Thus we obtain from $(\mathfrak{M}, G, \alpha)$ a canonically defined C^*-dynamical system $(\mathfrak{M}^c, G, \alpha|\mathfrak{M}^c)$. We shall study the relation between the W^*-crossed product $G \times_\alpha \mathfrak{M}$ and the C^*-crossed product $G \times_\alpha \mathfrak{M}^c$.

Recall from [8, §4] (cf. [5, 7.10.3]) that to each W^*-dynamical system $(\mathfrak{M}, G, \alpha)$ we can construct the dual system $(G \times_\alpha \mathfrak{M}, \hat{G}, \hat{\alpha})$. We may identify $\mathfrak{M}$ with the von Neumann subalgebra of $G \times_\alpha \mathfrak{M}$ consisting of the fixed points for $\hat{G}$ under

Received by the editors April 5, 1979 and, in revised form, August 17, 1979.
AMS (MOS) subject classifications (1970). Primary 46L05; Secondary 46L10.
Key words and phrases. C^*-dynamical systems, crossed products.

the dual action $\hat{\alpha}$. Moreover, if $t \to \lambda_t$ denotes the canonical unitary representation of G into $G \times_\alpha \mathfrak{M}$ (so that $G \times_\alpha \mathfrak{M}$ is generated by $\mathfrak{M}$ and λ_G) then $\hat{\alpha}_\sigma(\lambda_t) = (t, \sigma)\lambda_t$ for every t in G and σ in $\hat{G}$.

THEOREM. *Given a W^*-dynamical system $(\mathfrak{M}, G, \alpha)$ consider the dual system $(G \times_\alpha \mathfrak{M}, \hat{G}, \hat{\alpha})$. Let B denote the set of elements y in $(G \times_\alpha \mathfrak{M})^c$ for which the functions $t \to \lambda_t y$ and $t \to y\lambda_t$ are norm continuous. Then B is a $\hat{G}$-invariant C^*-algebra weakly dense in $G \times_\alpha \mathfrak{M}$ and B is canonically isomorphic to the C^*-crossed product $G \times_\alpha \mathfrak{M}^c$.*

PROOF. Set $\mathfrak{N} = G \times_\alpha \mathfrak{M}$. We know that $\mathfrak{N}^c$ is a $\hat{G}$-invariant C^*-algebra and it is elementary to check that the elements y in $\mathfrak{N}^c$ for which the function $t \to \lambda_t y$ is norm continuous form a norm closed right ideal R in $\mathfrak{N}^c$. Since $B = R^* \cap R$ we see that B by definition is a hereditary C^*-subalgebra of $\mathfrak{N}^c$. If $y \in \mathfrak{N}^c$ and $f, g \in L^1(G)$ then with $\lambda_f = \int \lambda_t f(t)\, dt$ we have $\lambda_f y \lambda_g \in B$ since translation is continuous in $L^1(G)$. From this it follows that B is weakly dense in $\mathfrak{N}^c$, hence in $\mathfrak{N}$. If $y \in B$ and $\sigma \in \hat{G}$ then

$$\|\lambda_t \hat{\alpha}_\sigma(y) - \hat{\alpha}_\sigma(y)\| = \|(\overline{t, \sigma})\hat{\alpha}_\sigma(\lambda_t y) - \hat{\alpha}_\sigma(y)\|$$
$$< \|\lambda_t y - y\| + \|(\overline{t, \sigma}) - 1\|\, \|y\|,$$

from which we infer that $\hat{\alpha}_\sigma(y) \in B$ so that B is $\hat{G}$-invariant.

We claim that B satisfies the following two conditions:

(*) The homomorphism $t \to \lambda_t$ takes G into the unitary group of the multiplier algebra $M(B)$ of B, see [5, 3.12] such that each function $t \to \lambda_t y$, $y \in B$, is norm continuous from G to B;

(**) There is a representation $\hat{\alpha}\colon \hat{G} \to \mathrm{Aut}(B)$ such that $(B, \hat{G}, \hat{\alpha})$ is a C^*-dynamical system and

$$\hat{\alpha}_\sigma(\lambda_t) = (t, \sigma)\lambda_t, \qquad t \in G, \sigma \in \hat{G}.$$

We have already verified condition (**) and of condition (*) we only need to verify that $\lambda_t \in M(B)$. But if $y \in B$ then $\lambda_t y \in \mathfrak{N}^c$ since $\hat{\alpha}_\sigma(\lambda_t y) = (t, \sigma)\lambda_t \hat{\alpha}_\sigma(y)$, and obviously the functions $s \to \lambda_{t+s} y$ and $s \to \lambda_t y \lambda_s$ are norm continuous so that $\lambda_t y \in B$. Using the involution we see that also $y\lambda_t \in B$, as desired. It follows from [4, 2.9] (cf. [5, 7.8.8]) that B is a G-product, i.e. B is the C^*-crossed product $G \times_\alpha A$, where $\alpha_t = \mathrm{Ad}\, \lambda_t$ and A is the C^*-subalgebra of $M(B)$ consisting of elements x that satisfy Landstad's conditions:

(i) $\hat{\alpha}_\sigma(x) = x$ for all σ in $\hat{G}$;

(ii) $x\lambda_f \in B$ and $\lambda_f x \in B$ for every f in $L^1(G)$;

(iii) The map $t \to \lambda_t x \lambda_{-t}$ $(= \alpha_t(x))$ is norm continuous on G.

Since $M(B) \subset \mathfrak{N}$ by [5, 3.12.5] we see from condition (i) that $A \subset \mathfrak{M}$, and from (iii) we further have $A \subset \mathfrak{M}^c$. Suppose now that $x \in \mathfrak{M}^c$ and take f in $L^1(G)$. Then $x\lambda_f \in \mathfrak{N}^c$ since

$$\|\hat{\alpha}_\sigma(x\lambda_f) - x\lambda_f\| = \|x\hat{\alpha}_\sigma(\lambda_f) - x\lambda_f\| < \|x\| \int |(t, \sigma) - 1|\,|f(t)|\, dt.$$

Actually $x\lambda_f \in B$ because

$$\|\lambda_t x\lambda_f - x\lambda_f\| = \|\alpha_t(x)\lambda_t\lambda_f - x\lambda_f\|$$
$$< \|\alpha_t(x) - x\| \, \|f\|_1 + \|x\| \int |f(s - t) - f(s)| \, ds.$$

Thus every element in $\mathfrak{M}^c$ satisfies condition (ii) (and, of course, also (i) and (iii)) so that $A = \mathfrak{M}^c$, and the proof is complete. Note that the dual C^*-system of $(\mathfrak{M}^c, G, \alpha)$ is $(B, \hat{G}, \hat{\alpha}|B)$.

COROLLARY 1. *Let $(\mathfrak{M}, G, \alpha)$ be a W^*-dynamical system where G is discrete, and consider the dual system $(G \times_\alpha \mathfrak{M}, \hat{G}, \hat{\alpha})$. Then $(G \times_\alpha \mathfrak{M})^c = G \times_\alpha \mathfrak{M}$, the latter taken as a C^*-crossed product.*

Let λ: $G \to L^2(G)$ denote the regular representation of the locally compact abelian group G, and note that with $\alpha_t = \operatorname{Ad} \lambda_t$ we have a W^*-dynamical system $(L^\infty(G), G, \alpha)$. Observe that since $(L^\infty(G))^c$ is the norm closure of elements of the form

$$\alpha_f(g) = g * f, \qquad g \in L^\infty(G), f \in L^1(G),$$

each of which belongs to the C^*-algebra $C_u^b(G)$ of bounded, uniformly continuous functions on G, we must have $(L^\infty(G))^c = C_u^b(G)$. It is well known that $G \times_\alpha L^\infty(G) = \mathfrak{B}(L^2(G))$ and that the dual action of $\hat{G}$ on $\mathfrak{B}(L^2(G))$ is given by $\hat{\alpha}_\sigma = \operatorname{Ad} \hat{\lambda}_\sigma$, where

$$(\hat{\lambda}_\sigma \xi)(t) = (t, \sigma)\xi(t), \qquad \xi \in L^2(G).$$

Applying the theorem we obtain

COROLLARY 2. *The set of elements x in $\mathfrak{B}(L^2(G))$ such that all functions $t \to \lambda_t x$, $t \to x\lambda_t$, and $\sigma \to \hat{\lambda}_\sigma x \hat{\lambda}_{-\sigma}$ are norm continuous on G and $\hat{G}$, respectively, is a C^*-algebra isomorphic to $G \times_\alpha C_u^b(G)$.*

If in the above we take G discrete, so that $C_u^b(G) = L^\infty(G)$, then Corollary 2 characterizes the C^*-crossed product $G \times_\alpha L^\infty(G)$ as the set of elements in $\mathfrak{B}(L^2(G))$ that transform continuously in norm under the action $\operatorname{Ad} \hat{\lambda}$ of $\hat{G}$. This result (with G and Γ in place of our $\hat{G}$ and G) is [6, 3.5], see also [3, 4.5].

Note that if translation is pointwise norm continuous on $L^\infty(G)$, i.e. if $C_u^b(G) = L^\infty(G)$, then G must be discrete. Indeed, let E be an open set in G and denote by p the corresponding characteristic function. Then either $\|\alpha_t(p) - p\|_\infty = 1$ or $\|\alpha_t(p) - p\|_\infty = 0$. Thus by our assumption there is a neighbourhood E_0 of 0 such that $\alpha_t(p) = p$, i.e. $E + t = E$ almost everywhere, for every t in E_0. But then $(E + t)\setminus\bar{E}$ is an open null set in G, and therefore empty; whence $E + t \subset \bar{E}$. It follows that $E + E_0 \subset \bar{E}$. Take a smaller neighbourhood E_1 of 0 such that $E_1 - E_1 \subset E_0$ and, to obtain a contradiction, assume that $s \in \bar{E}\setminus E$. Then $s - E_1$ intersects both E and $G\setminus\bar{E}$, i.e. $s - t_1 \in E$ and $s - t_2 \notin \bar{E}$ for t_1, t_2 in E_1. But now

$$\bar{E} \supset E + E_0 \ni (s - t_1) + (t_1 - t_2) = s - t_2 \notin \bar{E},$$

a contradiction. Thus $E = \bar{E}$, so that every open set is closed, i.e. G is discrete. Observe that the commutativity of G played no rôle in the argument.

We are now also in a position to show that, in the setting of Corollary 2, if $\sigma \to \hat{\lambda}_\sigma x \hat{\lambda}_{-\sigma}$, $\sigma \in \hat{G}$, is norm continuous for all x in $\mathfrak{B}(L^2(G))$ then G must be compact. Indeed, the weak closure of $L^1(G)$ in $\mathfrak{B}(L^2(G))$ is isomorphic with $L^\infty(\hat{G})$ and using Fourier transformation we see that the action of Ad $\hat{\lambda}$ on $L^\infty(\hat{G})$ is just translation. From the argument above continuity of translation implies that $\hat{G}$ is discrete, i.e. G is compact.

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