ON A CERTAIN $C^*$-CROSSED PRODUCT INSIDE A $W^*$-CROSSED PRODUCT

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Abstract. To each $W^*$-dynamical system $(\mathcal{M}, G, \alpha)$ corresponds canonically a $C^*$-dynamical system $(\mathcal{M}^c, G, \alpha|_{\mathcal{M}^c})$. We show that the $C^*$-crossed product $G \times_{\alpha} \mathcal{M}^c$ can be identified with a certain $C^*$-subalgebra of the $W^*$-crossed product $G \times_{\alpha} \mathcal{M}$.

The major part of the theory of noncommutative dynamical systems and their crossed products is Takesaki’s work; see e.g. [7] and [8]. An important contribution, however, was made by Landstad who in [1] characterized those operator algebras that are crossed products with a given locally compact group $G$. Landstad’s theory of $G$-products for abelian groups was exploited in [4] and [2] and we shall use it again to solve a problem arising from the difference between $C^*$- and $W^*$-crossed products.

A general exposition of noncommutative dynamical systems can be found in Chapters 7 and 8 of [5], but only the elementary parts of the theory will be needed here. Recall that a triple $(\mathcal{A}, G, \alpha)$ is called a $C^*$-dynamical system if $\mathcal{A}$ is a $C^*$-algebra and $\alpha$ is a representation of the locally compact abelian group $G$ as automorphisms on $\mathcal{A}$, such that each function $t \to \alpha_t(x), x \in \mathcal{A}$, is norm continuous. If $\mathcal{M}$ is a von Neumann algebra we define analogously a $W^*$-dynamical system $(\mathcal{M}, G, \alpha)$, but now only with the requirement that each function $t \to \alpha_t(x), x \in \mathcal{M}$, is $\sigma$-weakly continuous.

Given a $W^*$-dynamical system $(\mathcal{M}, G, \alpha)$ define $\mathcal{M}^c$ to be the set of elements $x$ in $\mathcal{M}$ for which the function $t \to \alpha_t(x)$ is norm continuous, see [5, 7.5.1]. Clearly $\mathcal{M}^c$ is a $G$-invariant $C^*$-subalgebra of $\mathcal{M}$ containing all elements of the form

$$\alpha_t(y) = \int \alpha_t(y)f(t) \, dt, \quad y \in \mathcal{M}, f \in L^1(G)$$

(since translation is continuous on $L^1(G)$). Using an approximate unit in $L^1(G)$ we see that $\mathcal{M}^c$ is in fact generated by elements $\alpha_t(y)$, and therefore $\sigma$-weakly dense in $\mathcal{M}$. Thus we obtain from $(\mathcal{M}, G, \alpha)$ a canonically defined $C^*$-dynamical system $(\mathcal{M}^c, G, \alpha|_{\mathcal{M}^c})$. We shall study the relation between the $W^*$-crossed product $G \times_{\alpha} \mathcal{M}$ and the $C^*$-crossed product $G \times_{\alpha} \mathcal{M}^c$.

Recall from [8, §4] (cf. [5, 7.10.3]) that to each $W^*$-dynamical system $(\mathcal{M}, G, \alpha)$ we can construct the dual system $(G \times_{\alpha} \mathcal{M}, \hat{G}, \hat{\alpha})$. We may identify $\mathcal{M}$ with the von Neumann subalgebra of $G \times_{\alpha} \mathcal{M}$ consisting of the fixed points for $\hat{G}$ under

Received by the editors April 5, 1979 and, in revised form, August 17, 1979.


Key words and phrases. $C^*$-dynamical systems, crossed products.
the dual action $\hat{\alpha}$. Moreover, if $t \mapsto \lambda_t$ denotes the canonical unitary representation of $G$ into $G \rtimes \mathbb{R}$ (so that $G \rtimes \mathbb{R}$ is generated by $\mathbb{R}$ and $\lambda_G$) then $\hat{\alpha}_a(\lambda_t) = (t, \sigma)\lambda_t$ for every $t$ in $G$ and $\sigma$ in $\hat{G}$.

**Theorem.** Given a $\mathbb{W}^*$-dynamical system $(\mathbb{M}, G, \alpha)$ consider the dual system $(G \times_a \mathbb{M}, G, \hat{\alpha})$. Let $B$ denote the set of elements $y$ in $(G \times_a \mathbb{M})^c$ for which the functions $t \mapsto \lambda_t y$ and $t \mapsto y\lambda_t$ are norm continuous. Then $B$ is a $\hat{G}$-invariant $\mathbb{C}^*$-algebra weakly dense in $G \times_a \mathbb{M}$ and $B$ is canonically isomorphic to the $\mathbb{C}^*$-crossed product $G \times_a \mathbb{M}^c$.

**Proof.** Set $\mathbb{N} = G \times_a \mathbb{M}$. We know that $\mathbb{N}^c$ is a $\hat{G}$-invariant $\mathbb{C}^*$-algebra and it is elementary to check that the elements $y$ in $\mathbb{N}^c$ for which the function $t \mapsto \lambda_t y$ is norm continuous form a norm closed right ideal $R$ in $\mathbb{N}$. Since $B = R^* \cap R$ we see that $B$ by definition is a hereditary $\mathbb{C}^*$-subalgebra of $\mathbb{N}$. If $y \in \mathbb{N}$ and $f, g \in L^1(G)$ then with $X_y = f \lambda_t y$ we have $X_y A \lambda_x B \in B$ since translation is continuous in $L^1(G)$. From this it follows that $B$ is weakly dense in $\mathbb{N}$, hence in $\mathbb{N}$. If $y \in B$ and $\sigma \in \hat{G}$ then

$$\|\lambda_s \hat{\alpha}_\sigma(y) - \hat{\alpha}_\sigma(y)\| = \|\lambda_s y - y\| \leq \|\lambda_s y - y\| + \|\lambda_s y - 1\||y||,$$

from which we infer that $\hat{\alpha}_\sigma(y) \in B$ so that $B$ is $\hat{G}$-invariant.

We claim that $B$ satisfies the following two conditions:

(1) The homomorphism $t \mapsto \lambda_t$ takes $G$ into the unitary group of the multiplier algebra $M(B)$ of $B$, see [5, 3.12] such that each function $t \mapsto \lambda_t y$, $y \in B$, is norm continuous from $G$ to $B$;

(2) There is a representation $\hat{\alpha}: \hat{G} \to \text{Aut}(B)$ such that $(B, \hat{G}, \hat{\alpha})$ is a $\mathbb{C}^*$-dynamical system and

$$\hat{\alpha}_\sigma(\lambda_t) = (t, \sigma)\lambda_t, \quad t \in G, \sigma \in \hat{G}.$$

We have already verified condition (2) and of condition (1) we only need to verify that $\hat{\alpha}_\sigma \in M(B)$. But if $y \in B$ then $\lambda_t y \in \mathbb{N}^c$ since $\hat{\alpha}_\sigma(\lambda_t y) = (t, \sigma)\lambda_t y$, and obviously the functions $s \mapsto \lambda_s y$ and $s \mapsto \lambda_s y \lambda_t$ are norm continuous so that $\lambda_t y \in B$. Using the involution we see that also $y\lambda_t \in B$, as desired. It follows from [4, 2.9] (cf. [5, 7.8.8]) that $B$ is a $\mathbb{C}^*$-product, i.e. $B$ is the $\mathbb{C}^*$-crossed product $G \times_a A$, where $\alpha_t = \text{Ad} \lambda_t$ and $A$ is the $\mathbb{C}^*$-subalgebra of $M(B)$ consisting of elements $x$ that satisfy Landstad's conditions:

(i) $\hat{\alpha}_\sigma(x) = x$ for all $\sigma$ in $\hat{G}$;

(ii) $x\lambda_t y \in B$ and $\lambda_t x y \in B$ for every $f$ in $L^1(G)$;

(iii) The map $t \mapsto x\lambda_t y$ (or $\alpha_t(x)$) is norm continuous on $G$.

Since $M(B) \subset \mathbb{N}$ by [5, 3.12.5] we see from condition (i) that $A \subset \mathbb{N}$, and from (iii) we further have $A \subset \mathbb{N}^c$. Suppose now that $x \in \mathbb{N}^c$ and take $f$ in $L^1(G)$. Then $x\lambda_t \in \mathbb{N}^c$ since

$$\|x\lambda_t - x\lambda_s\| = \|x\lambda_t - x\lambda_s\| < \|x\| \int \|t, \sigma\| |f(t)| dt.$$
Actually $x\lambda_{\gamma} \in B$ because 
$$
\|\lambda_{\gamma} x\lambda_{\gamma} - x\lambda_{\gamma}\| = \|\alpha_\gamma(x)\lambda_{\gamma} - x\lambda_{\gamma}\| 
< \|\alpha_\gamma(x) - x\| \|f\|_1 + \|x\| \int |f(s - t) - f(s)| \, ds.
$$

Thus every element in $\mathcal{R}^c$ satisfies condition (ii) (and, of course, also (i) and (iii)) so that $A = \mathcal{R}^c$, and the proof is complete. Note that the dual $C^*$-system of $(\mathcal{R}^c, G, \alpha)$ is $(B, \hat{G}, \hat{\alpha} | B)$.

**Corollary 1.** Let $(\mathcal{R}, G, \alpha)$ be a $\mathcal{W}^*$-dynamical system where $G$ is discrete, and consider the dual system $(G \times_\alpha \mathcal{R}, G, \hat{\alpha})$. Then $(G \times_\alpha \mathcal{R})^c = G \times_\alpha \mathcal{R}$, the latter taken as a $C^*$-crossed product.

Let $\lambda : G \to L^2(G)$ denote the regular representation of the locally compact abelian group $G$, and note that with $\alpha_t = \text{Ad} \lambda_t$, we have a $\mathcal{W}^*$-dynamical system $(L^\infty(G), G, \alpha)$. Observe that since $(L^\infty(G))^c$ is the norm closure of elements of the form 
$$
\alpha_t(g) = g \ast f, \quad g \in L^\infty(G), f \in L^1(G),
$$
each of which belongs to the $C^*$-algebra $C_\alpha^b(G)$ of bounded, uniformly continuous functions on $G$, we must have $(L^\infty(G))^c = C_\alpha^b(G)$. It is well known that $G \times_\alpha L^\infty(G) = \mathfrak{B}(L^2(G))$ and that the dual action of $\hat{G}$ on $\mathfrak{B}(L^2(G))$ is given by $\hat{\alpha}_t = \text{Ad} \hat{\lambda}_t$, where 
$$
(\hat{\lambda}_t \xi)(t) = (t, \sigma) \xi(t), \quad \xi \in L^2(G).
$$

Applying the theorem we obtain

**Corollary 2.** The set of elements $x$ in $\mathfrak{B}(L^2(G))$ such that all functions $t \to \lambda_{\gamma} x$, $t \to x\lambda_{\gamma}$, and $\sigma \to \hat{\lambda}_\sigma x\hat{\lambda}_{-\sigma}$ are norm continuous on $G$ and $\hat{G}$, respectively, is a $C^*$-algebra isomorphic to $G \times_\alpha C_\alpha^b(G)$.

If in the above we take $G$ discrete, so that $C_\alpha^b(G) = L^\infty(G)$, then Corollary 2 characterizes the $C^*$-crossed product $G \times_\alpha L^\infty(G)$ as the set of elements in $\mathfrak{B}(L^2(G))$ that transform continuously in norm under the action $\text{Ad} \hat{\lambda}$ of $\hat{G}$. This result (with $G$ and $\Gamma$ in place of our $\hat{G}$ and $G$) is [6, 3.5], see also [3, 4.5].

Note that if translation is pointwise norm continuous on $L^\infty(G)$, i.e. if $C_\alpha^b(G) = L^\infty(G)$, then $G$ must be discrete. Indeed, let $E$ be an open set in $G$ and denote by $p$ the corresponding characteristic function. Then either $\|\alpha_t(p) - p\|_\infty = 1$ or $\|\alpha_t(p) - p\|_\infty = 0$. Thus by our assumption there is a neighbourhood $E_0$ of 0 such that $\alpha_t(p) = p$, i.e. $E + t = E$ almost everywhere, for every $t$ in $E_0$. But then $(E + t) \Delta \overline{E}$ is an open null set in $G$, and therefore empty; whence $E + t \subseteq \overline{E}$. It follows that $E + E_0 \subseteq \overline{E}$. Take a smaller neighbourhood $E_1$ of 0 such that $E_1 - E_1 \subseteq E_0$ and, to obtain a contradiction, assume that $s \in \overline{E} \setminus E$. Then $s - E_1$ intersects both $E$ and $G \setminus \overline{E}$, i.e. $s - t_1 \in E$ and $s - t_2 \not\in \overline{E}$ for $t_1, t_2$ in $E_1$. But now 
$$
\overline{E} \supset E + E_0 \supset (s - t_1) + (t_1 - t_2) = s - t_2 \not\in \overline{E},
$$
a contradiction. Thus $E = \overline{E}$, so that every open set is closed, i.e. $G$ is discrete. Observe that the commutativity of $G$ played no rôle in the argument.
We are now also in a position to show that, in the setting of Corollary 2, if 
\[ \sigma \mapsto \hat{\lambda}_\sigma x \hat{\lambda}_{-\sigma}, \sigma \in \hat{G}, \] is norm continuous for all \( x \) in \( \mathfrak{B}(L^2(G)) \) then \( G \) must be compact. Indeed, the weak closure of \( L^1(G) \) in \( \mathfrak{B}(L^2(G)) \) is isomorphic with \( L^\infty(\hat{G}) \) and using Fourier transformation we see that the action of \( \text{Ad} \hat{\lambda} \) on \( L^\infty(\hat{G}) \) is just translation. From the argument above continuity of translation implies that \( \hat{G} \) is discrete, i.e. \( G \) is compact.

References


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