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A CHARACTERIZATION OF THE RANGE OF A BOUNDED LINEAR TRANSFORMATION IN HILBERT SPACE

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ABSTRACT. It is a theorem of Smul'jan and Mac Nerney that for B a bounded linear transformation from a complete complex inner product space $\{S, (\cdot, \cdot)\}$ to S, with adjoint transformation B^*, $B(S)$ is the set of all z in S for which there is a nonnegative number b such that for all x in S, $|(z, x)|^2 \leq b\|B^*x\|^2$, in which case if w is that point of $(\ker B)^\perp$ such that $Bw = z$ then the least such b is $\|w\|^2$. This paper provides another description of $B(S)$ and formula for $\|w\|^2$.

Throughout this report, it will be supposed that $\{S, (\cdot, \cdot)\}$ is a complete complex inner product space with norm $\|\cdot\|$ and that B is a bounded linear transformation from S to S with operator norm $\|B\| \neq 0$. The notation of the paper is consistent with that of [2].

THEOREM 1. *Suppose* $0 \leq B \leq 1$. *Then* $B(S)$ *is the set of all points* y *in* S *for which the series* $\sum_{p=0}^{\infty}(1 - B)^p y$ *converges with respect to* $\|\cdot\|$, *in which case the series has limit that point* x_1 *of* $(\ker B)^\perp$ *such that* $Bx_1 = y$.

PROOF. Suppose P is the orthogonal projection of S onto $(\ker B)^\perp$ and E_λ is the spectral resolution of B, with $E_{0-} = 0$, E_1 the identity on S, and E_λ right-continuous at each λ. Supposing $n \geq 0$ and x in S, one has

$$x - \sum_{p=0}^{n} (1 - B)^p Bx = (1 - B)^{n+1}x. \qquad (*)$$

Thus, with $x_1 = Px$,

$$\left\| x_1 - \sum_{p=0}^{n} (1 - B)^p Bx_1 \right\|^2 = \|(1 - B)^{n+1}x_1\|^2 = \int_{0-}^{1} (1 - \lambda)^{2(n+1)} d(E_\lambda x_1, x_1).$$

Since P commutes with B and, hence, with E_0, $E_0 x_1$ is in $\ker(B) \cap \ker(B)^\perp = \{0\}$. Hence, the measure of $\{0\}$ with respect to the measure $(E(\cdot)x_1, x_1)$ is 0. In view of this and the uniform boundedness and convergence to 0 on $(0, 1]$ of $(1 - \lambda)^{2(n+1)}$, the integral $\int_{0-}^{1}(1 - \lambda)^{2(n+1)}d(E_\lambda x_1, x_1) \to 0$ as $n \to \infty$. Hence, $\sum_{p=0}^{\infty}(1 - B)^p Bx$ has limit x_1, the unique pre-image of Bx in $\ker(B)^\perp$; in other words, the vector of least norm mapped by B onto Bx.

Conversely, if $\sum_{p=0}^{\infty}(1 - B)^p y = z$ then $(1 - B)z = z - y$. Thus $y = Bz$. One may note that Theorem 1 is a special case of the following. In case E is a

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topological vector space and B a continuous linear transformation on E such that for x in E $B^n x \to 0$, then y is in $(1 - B)(E)$ if only if $\Sigma_0^\infty B^p y$ is convergent.

THEOREM 2. *The range $B(S)$ of B is the set of all y in S for which the series*

$$\|B\|^{-2} \sum_{p=0}^{\infty} \|(1 - BB^*/\|B\|^2)^{p/2} y\|^2$$

converges, in which case if x_1 is that point of $(\ker B)^\perp$ *such that $Bx_1 = y$ the series has limit x_1.*

PROOF. Suppose, initially, that $0 \le B \le 1$. Then, with the notation of the argument of Theorem 1, one has, upon replacing B by B^2 in $(*)$ and computing the inner product of both sides with x_1,

$$\|x_1\|^2 - \sum_{p=0}^{n} \|(1 - B^2)^{p/2} B x_1\|^2 = \left((1 - B^2)^{n+1} x_1, x_1\right)$$

$$= \int_{0-}^{1} (1 - \lambda^2)^{n+1} d(E_\lambda x_1, x_1),$$

the integral tending to 0 by an argument similar to that in Theorem 1.

Conversely, suppose that $\Sigma_{p=0}^\infty \|(1 - B^2)^{p/2} y\|^2 < \infty$. Since, with $Py = y_1$ and $(1 - P)y = y_2$,

$$\sum_{p=0}^{n} \|(1 - B)^{p/2} y\|^2 = \sum_{p=0}^{n} \left\{\left((1 - B^2)^p y_1, y_1\right) + \left((1 - B^2)^p y_2, y_2\right)\right\} \ge \sum_{p=0}^{n} \|y_2\|^2,$$

y is in $\ker(B)^\perp$.

Since $(\Sigma_{p=n}^m (1 - \lambda^2)^p \lambda)^2 \le \Sigma_{p=n}^m (1 - \lambda^2)^p$ on $[0, 1]$, the spectral theorem gives

$$\left\| \sum_{p=n}^{m} (1 - B^2)^p B y \right\|^2 \le \sum_{p=n}^{m} \left((1 - B^2)^p y, y\right) = \sum_{p=n}^{m} \|(1 - B^2)^{p/2} y\|^2.$$

Therefore, $\Sigma_{p=0}^\infty (1 - B^2)^p B y = z$ exists in S. Then $(1 - B^2)z = z - By$ so that $y - Bz$ is in $\ker(B) \cap \ker(B)^\perp$ and $y = Bz$.

In the more general case, one has the polar decomposition $B = (BB^*)^{1/2} U$ where U is a partial isometry with initial space $\ker(B)^\perp$. If z is in $B(S)$, then $z = Bw$ for a unique w in $\ker(B)^\perp$. Hence, $z = (BB^*)^{1/2} r$ with $r = Uw$ and $\|r\| = \|w\|$. The theorem follows from an application of the argument for the case $0 \le B \le 1$ to $(BB^*)^{1/2}/\|B\|$. Indeed, the observation that for B an operator between Hilbert spaces the range of B is the range of $(BB^*)^{1/2}$ provides a two-space analogue of the theorem.

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