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# NORMALITY CAN BE RELAXED IN THE ASYMPTOTIC FUGLEDE-PUTNAM THEOREM

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ABSTRACT. The original form of the Fuglede-Putnam theorem states that the operator equation $AX = XB$ implies $A^*X = XB^*$ when $A$ and $B$ are normal. In our previous paper we have relaxed the normality in the hypotheses on $A$ and $B$ as follows: *if $A$ and $B^*$ are subnormal and if $X$ is an operator such that $AX = XB$, then $A^*X = XB^*$*. We shall show asymptotic versions of this generalized Fuglede-Putnam theorem; these results are also extensions of results of Moore and Rogers.

1. An operator means a bounded linear operator on a complex Hilbert space. An operator $T$ is called *quasinormal* if $T$ commutes with $T^*T$, *subnormal* if $T$ has a normal extension and *hyponormal* if $T^*T \geq TT^*$. The class of subnormal operators properly contains the class of quasinormal operators and is properly contained in the class of hyponormal operators [5, Problem 160, p. 101]. We have shown Theorem A [3, Theorem 1] as an extension of the Fuglede-Putnam theorem by an easy calculation.

THEOREM A [3]. *If $A$ and $B^*$ are subnormal and if $X$ is an operator such that $AX = XB$, then $A^*X = XB^*$.*

On the other hand, using techniques inspired by those of Rosenblum [9] and also employing Berberian's trick [1], Moore [6] shows the original asymptotic version of the Fuglede-Putnam theorem as follows.

THEOREM B [6]. *Let $A$ and $B$ be normal. For each $\varepsilon > 0$, there exists $\delta$ such that $\|X\| \leq 1$ and $\|AX - XB\| < \delta$ imply $\|A^*X - XB^*\| < \varepsilon$.*

Moreover, scrutinizing Moore's proof, Rogers shows the following Theorems C and D analogous to Moore's in which the norm topology in Theorem B can be replaced by the strong or weak operator topology.

THEOREM C [8]. *If $A$ and $B$ are normal operators and if $E$ is a neighborhood of 0 in the strong [resp., weak] operator topology, then there is a neighborhood $D$ of 0 in the same topology such that the conditions $\|X\| \leq 1$ and $AX - XB \in D$ imply $A^*X - XB^* \in E$.*

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THEOREM D [8]. *Let $\psi$ be a complex-valued continuous function on the union of the spectra of the normal operators $A$ and $B$. For each neighborhood $E$ of* 0 *in the strong [resp., weak] operator topology there is a neighborhood $D$ of* 0 *in the same topology such that the conditions $\|X\| \leq 1$ and $AX - XB \in D$ imply $\psi(A)X - X\psi(B) \in E$.*

In this paper, combining the idea used to show Theorem A with the techniques used in proving Theorems B, C and D, we shall show Theorems 1 and 2. These results are extensions of Theorems B, C and D and are asymptotic versions of Theorem A. Finally we shall pose an open problem with respect to Theorems 1 and 2.

**2.** First we show Theorem 1, which is an asymptotic version of the generalized Fuglede-Putnam theorem and extends Theorems B and C.

THEOREM 1. *Let $A$ and $B^*$ be subnormal operators. If $E$ is a neighborhood of* 0 *in the uniform [resp. strong operator, weak operator] topology, then there is a neigborhood $D$ of* 0 *in the same topology such that the conditions $\|X\| \leq 1$ and $AX - XB \in D$ imply $A^*X - XB^* \in E$.*

PROOF. The idea [Added in proof [3], Another proof of Theorem 1], together with the techniques in [6] and [8], yields the proof of the result. A normal extension $N_A$ of $A$ on the Hilbert space $H$ is given by

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

acting on the Hilbert space $H \oplus H$ whose restriction to $H \oplus \{0\}$ is $A$ [4] and a normal one $N_{B^*}$ of $B^*$ on $H$ is also given by

$$N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

acting on $H \oplus H$. We define the subset $\tilde{E}$ as follows:

$$\tilde{E} = \left\{ \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} : Y_k \text{ in } E \ (k = 1, 2, 3, 4) \right\}$$

in the set of operators on $H \oplus H$. Then $\tilde{E}$ turns out to be a neighborhood of 0 on $H \oplus H$ in the same topology (uniform, strong or weak) that $E$ is on $H$. By Theorems B and C, there exists a neighborhood $\tilde{D}$ of 0 on $H \oplus H$ such that any operator $\tilde{X}$ on $H \oplus H$ with $\|\tilde{X}\| \leq 1$ and $N_A\tilde{X} - \tilde{X}N_{B^*}^* \in \tilde{D}$ has $N_A^*\tilde{X} - \tilde{X}N_{B^*} \in \tilde{E}$. Define $D = \{Y : \left(\begin{smallmatrix} Y & 0 \\ 0 & 0 \end{smallmatrix}\right) \text{ is in } \tilde{D}\}$. Then this set $D$ turns out to be a neighborhood of 0 on $H$ in the same topology that $\tilde{D}$ is on $H \oplus H$. Assume $X$ is an operator on $H$ with $\|X\| \leq 1$ and $AX - XB = Y$ in $D$. Put $\tilde{X} = \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $H \oplus H$. Then $\|\tilde{X}\| \leq 1$ and

$$N_A\tilde{X} - \tilde{X}N_{B^*}^* = \begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix}$$

is in $\tilde{D}$. Hence we have

$$N_A^*\tilde{X} - \tilde{X}N_{B^*} = \begin{pmatrix} A^*X - XB^* & -XB_{12} \\ A_{12}^*X & 0 \end{pmatrix}$$

is in $\tilde{E}$, which implies that $A^*X - XB^*$ is in $E$, $-XB_{12}$ and $A_{12}^*X$ are also in $E$, so the proof is complete.

In [6, Corollary 2] Moore shows Theorem D in the case of the uniform topology. Hence the following Theorem 2 is an extension of Theorem D and [6, Corollary 2].

THEOREM 2. *Let $A$ and $B^*$ be subnormal operators and $\psi$ be a complex-valued continuous function on the union of the spectra of $A$ and $B$. For each neighborhood $E$ of 0 in the uniform [resp. strong operator, weak operator] topology, there is a neighborhood $D$ of 0 in the same topology such that the conditions $\|X\| \leq 1$ and $AX - XB \in D$ imply $\psi(A)X - X\psi(B) + \phi \in E$, where $\phi$ is a function of $A, B, \psi$ and $X$. In addition, $\phi = 0$ holds under any one of the following hypotheses:*

(1) *$A$ and $B$ are both normal operators,*

(2) *$\psi$ is a function of $z$ or $\psi$ is a function of $\bar{z}$.*

PROOF. The idea of the proof is similar to the one of Theorem 1. We retain the notations of Theorem 1. By Theorem D and [6, Corollary 2], there exists a neighborhood $\tilde{D}$ of 0 on $H \oplus H$ such that any operator $\tilde{X}$ on $H \oplus H$ with $\|\tilde{X}\| \leq 1$, and $N_A\tilde{X} - \tilde{X}N_{B^*}^*$ in $\tilde{D}$ has $\psi(N_A)\tilde{X} - \tilde{X}\psi(N_{B^*}^*)$ in $\tilde{E}$. Define $D = \{Y: \left(\begin{smallmatrix} Y & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in $\tilde{D}\}$; then this set $D$ turns out to be a neighborhood of 0 on $H$ in the same topology that $\tilde{D}$ is on $H \oplus H$ as stated in the proof of Theorem 1. Assume $X$ is an operator on $H$ with $\|X\| \leq 1$ and $AX - XB = Y$ in $D$. Put $\tilde{X} = \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $H \oplus H$. Then $\|\tilde{X}\| \leq 1$ and

$$N_A\tilde{X} - \tilde{X}N_{B^*}^* = \begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix}$$

is in $\tilde{D}$. Hence we have

$$\psi(X_A)\tilde{X} - \tilde{X}\psi(N_{B^*}^*) = \begin{pmatrix} \psi(A)X - X\psi(B) + \phi & * \\ ** & 0 \end{pmatrix}$$

is in $\tilde{E}$, which implies $\psi(A)X - X\psi(B) + \phi$ is in $E$, where $\phi$ is a function of $A$, $B$, $\psi$ and $X$. The proof in the case (1) of $\phi = 0$ follows from Theorem D and [6, Corollary 2] and for the proof in the case (2) of $\phi = 0$, it is sufficient to remark that a continuous function of a triangular operator matrix is also one of the same type. Hence the proof is complete.

REMARK 1. In Theorem 2, $\phi$ can be considered as a "*perturbed term*" *which measures the deviation of subnormality from normality*. If $\psi(z) = \bar{z}$, then $\phi = 0$ by (2) of Theorem 2, and this is just Theorem 1.

REMARK 2. In Theorems 1 and 2 we cannot replace the *subnormality* in the hypotheses on $A$ and $B^*$ by the *subnormality* on $A$ and $B$. Assume we could; then similarity for $A$ and $B$ would imply unitary equivalence by [3, Corollary 1]. But that is impossible because there exists a counterexample as follows: there exist two subnormal operators that are similar but not unitarily equivalent [5, Solution 156]. *Hence we remark that Theorems* 1 *and* 2 *do not involve symmetric hypotheses on $A$ and $B$, but rather on $A$ and $B^*$.* In view of this, it is natural and reasonable in Theorems B, C and D to interpret the hypothesis of normality of $A$ and $B$ as that of normality of $A$ and $B^*$.

Finally we pose the following open question.

*Open question*. It is natural to ask whether subnormality can be replaced by hyponormality in Theorems 1 and 2. Modest results are cited in [**10**, Proposition], [**2**, Theorem] and [**3**, Corollary 2]. But we cannot solve this problem.

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