THE ZYGMUND CONDITION FOR BLOCH FUNCTIONS
IN THE BALL IN $C^n$

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Abstract. In this paper we prove the equivalence of the Bloch condition for a
holomorphic function $f$ on the ball $B_n$ with the Zygmund second difference
condition for a suitable primitive $F$ of $f$.

Introduction. If $B_1$ is the open unit disc in the complex plane and $f$ is holomor-
phic on $B_1$, we say that $f$ is a Bloch function if there exists a positive number $M$
such that

$$|f'(z)|(1 - |z|^2) < M$$

for all $z \in B_1$. When equipped with an appropriate norm the linear space of Bloch
functions becomes a nonseparable Banach space. There are many equivalent
conditions that a function can satisfy to be a Bloch function (see Pommerenke [2]
or Cima [1]). In a recent thesis, Richard Timoney [4] has done an exhaustive study
of properties of Bloch functions on domains in $C^n$. In particular his work includes
the theory of Bloch functions on the ball

$$B_n = \left\{ z \in C^n : \|z\| = \sqrt{|z|^2} < 1 \right\}.$$

He has shown that all the known characterizations, save two, that are equivalent
for the case of $n = 1$ are valid for $n > 1$. One of these two characterizations is the
second difference condition of Zygmund [5]. We will establish the equivalence of
this condition for the $B_n$ case.

1. Preliminaries. Assume $f$ is a holomorphic function of $B_n \to C$. For $u$ and $v$
vectors in $C^n$, $z \in B_n$ and $\langle u, v \rangle = \Sigma_{j=1}^n u_j \bar{v}_j$ the Bergman metric is given by

$$H_z(u, \bar{v}) = \left( \frac{n + 1}{2} \right) \left[ \frac{(1 - \|z\|^2)\langle u, \bar{v} \rangle + \langle u, \bar{x} \rangle \langle z, \bar{v} \rangle}{(1 - \|z\|^2)^2} \right].$$

For each $z \in B_n$ define

$$Q_f(z) \equiv \sup \left\{ |(\nabla_x f)(x)| / H_z(x, \bar{x})^{1/2}; \ x \in C^n, x \neq 0 \right\}$$

where $(\nabla_x f)(x) = \Sigma_{j=1}^n (\partial f / \partial z_j)(x) x_j$. 

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**Definition 1.** A holomorphic function $f: B_n \to C$ is called a Bloch function if
sup\{\frac{\beta}{|z|} : z \in B_n\} < \infty.

In considering this definition, a certain amount of pathology immediately enters. In fact one observes by studying the metric that if $f$ is a Bloch function, then the growth of its directional derivative in the radial direction is $O((1 - \|z\|^2)^{-1})$, whereas the growth in directions orthogonal to the radial direction is only $O((1 - \|z\|^2)^{-1/2})$. Timoney [4] shows that the Bloch condition is equivalent to the following condition:

$$\sup\{\|\nabla f\|(1 - \|z\|^2) : z \in B_n\} < \infty. \quad (1.1)$$

Assume $F$ is in the ball algebra of $B_1$, i.e., $F$ is continuous on $B_1$ and holomorphic in $B_1$. Further assume that the boundary values of $F$ satisfy

$$F(e^{i(e+h)}) + F(e^{i(-h)}) - 2F(e^{i(t)}) < A|h| \quad (1.2)$$

for all real numbers $h$ and some positive number $A$, independent of $\theta$. In [5] it was shown that a function $f$, holomorphic on $B_1$, satisfies the Bloch condition if and only if its primitive $F(z) \equiv \int_0^f f(\xi) d\xi$ is in the ball algebra of $B_1$ and satisfies condition (1.2).

We consider in this note $C^1$ curves $\gamma$ mapping $R \to \partial B_n$ such that $|\gamma'(t)| = 1$ for all $t$. We refer to these as normalized $C^1$ curves. With this class of curves in mind we make the following definition.

**Definition 1.2.** Let $F$ be in the ball algebra of $B_n$. We say that $F$ satisfies condition $\Lambda_\alpha(\partial B_n)$ if there exists a positive number $A$ such that for all normalized $C^1$ curves $\gamma$ in $B_n$,

$$|F(\gamma(t + h)) + F(\gamma(t - h)) - 2F(\gamma(t))| < A|h|$$

for all $t$ and $h$ in $R$.

Finally, if $f$ is holomorphic on $B_n$ with expansion in terms of homogeneous polynomials given by $f(z) = \sum_{k=0}^{\infty} F_k(z)$ define the radial derivative $Rf$ of $f$ by the formula

$$\Re f(z) \equiv \sum_{k=1}^{\infty} kF_k(z).$$

**2. The principal result.** If we are given a function $f$ holomorphic on $B_n$, define a function

$$F(z) = (\Re f)(z) = \int_0^1 f(tz) \, dt.$$ 

One checks that $\Re (\Re f)(z) = \Re F(z) = f(z) - F(z)$.

**Theorem 1.** A function $f$ holomorphic on $B_n$ is in the Bloch space of $B_n$ if and only if $\Re f$ satisfies the $\Lambda_\alpha(\partial B_n)$ condition.

**Proof.** Assume first that $f$ is a Bloch function. For $a \in \partial B_n$ the slice functions are defined by $f_a(\lambda) = f(\lambda a)$, $\lambda \in B_1$. Since

$$\Re f_a(\lambda) = \frac{1}{\lambda} \int_0^\lambda f_a(\xi) \, d\xi \quad (2.1)$$
we see that each $F_a$ is in $\Lambda_\omega(\partial B_1)$. Also each member of the family $\{F_a; a \in \partial B_n\}$ has its oscillation $\omega(F_a, \delta) = O(\delta \log \delta)$, uniformly in $a$. $F$ can be extended to a function on $\overline{B}_n$ by using the values on the slices. Now with $b \in \partial B_n$, $0 < r < 1$,

$$|F(a) - F(b)| < M(1 - r) \log(1 - r) + |F_a(r) - F_b(r)|.$$ 

This shows that $F$ is continuous at $a$ and hence is in the ball algebra.

Now fix a normalized $C^1$ curve $\gamma$ with range in $\partial B_n$ and let $h > 0$ be given. If $g(t)$ is any function defined on $\mathbb{R}$ set $A_g(t) = g(t + h) - g(t)$. With $r = 1 - h$ we write

$$F(\gamma(t)) = (F(\gamma(t)) - F(\gamma(t))) + (F(\gamma(t)))$$

and show that $A^2$ of each expression in parentheses is $O(h)$, independent of $t$. Since $|f(z)| = O(\log(1 - |z|))$ one easily verifies that

$$F(\gamma(t)) - F(\gamma(t)) = (1 - r)f(\gamma(t)) + \int_0^1 (1 - s) \nabla f(\gamma(t)) \circ \gamma(t) \, ds.$$ 

The integral in this equality is $O(h)$. Also

$$A^2 f(\gamma(t)) = \int_0^h \nabla f(\gamma(t)) \circ \gamma'(t) = O(1)$$

where $\gamma(p) = \gamma(t + p)$. Hence

$$A^2[F(\gamma(t)) - F(\gamma(t))] = O(h)$$

uniformly in $t$. The expression $A^2 F(\gamma(t))$ involves three terms:

$$r \int_0^h (\gamma'(t + p) - \gamma'(t)) \circ \nabla F(\gamma(t + p)) \, dp, \quad (2.2)$$

$$r \int_0^h \gamma'(t) \circ \nabla F(\gamma(t + p) - \nabla F(\gamma(t - p))) \, dp, \quad (2.3)$$

$$r \int_0^h (\gamma'(t - \gamma'(t - p)) \circ \nabla F(\gamma(t - p))) \, dp. \quad (2.4)$$

By the definition

$$\left| \frac{\partial F}{\partial z_j}(\gamma(t)) \right| = \left| r \int_0^1 \nabla f(r\gamma(t)) \circ \gamma'(t) u \, du \right| < M \cdot | \log h|.$$ 

Thus, expressions (2.2) and (2.4) are $O(1)$ uniformly in $t$. Similarly

$$\| \nabla F(\gamma(t)) - \nabla F(\gamma(s)) \| = O(\log(1 - r))$$

and hence (2.3) is $O(1)$.

For the converse we observe that for each $0 < \alpha < 1$, the space $\Lambda_\alpha(B_1)$ of functions in the ball algebra with boundary values in the Lip $\alpha$ space contains $\Lambda_\omega(B_1)$. Further, bounded subsets of $\Lambda_\omega$ are bounded in $\Lambda_\alpha$. Fix $z = re_1$ in $B_n$ with $\|e_1\| = 1$, $0 < r < 1$, and let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $C^n$. Let $D_j$ be the derivative in the $e_j$ direction. We can apply a result of Rudin [3] to draw the following conclusions. Since $\{F_a\}$ is a norm-bounded subset of $\Lambda_{1/2}(B_1)$,

$$(\Re F)(z) = O((1 - \|z\|)^{-1/2})$$

and

$$D_j f(z) = O((1 - \|z\|)^{-1}), \quad 2 < j < n.$$
Since $F_a(r) + (\Re F)_a(r) = f_a(r)$ we apply the one variable result to conclude
\[ D_1 f(z) = Df_a(r) = O((1 - \|z\|)^{-1}). \]

The estimates are uniform.

The referee has pointed out that our proof yields the following equivalences.

**Proposition 1.** A homomorphic function $f: B_n \to C$ is a Bloch function if and only if the slice functions $F_a = (\Re f)_a, a \in \partial B_n$, are uniformly bounded in $\Lambda_a(\partial B_1)$.

**Proof.** From [4] a function $f: B_n \to C$ is a Bloch function if and only if
\[ \sup_{z \in B_n} |(\Re f)(z)|(1 - \|z\|^2) < \infty. \quad (2.5) \]
The functions $(\Re f)_a, a \in \partial B_n$, are uniformly bounded in $\Lambda_a(\partial B_1)$ if and only if
\[ \sup_{|z| < 1} |[(\Re f)_a]'(z)|(1 - |z|^2) < \infty. \quad (2.6) \]

A computation with (2.1) shows that
\[ [(\Re f)_a]''(z) = \frac{1}{z^2} (\Re f)_a(z) - \frac{2}{z^2} [f_a(z) - (\Re f)_a(z)] \]
\[ = \frac{1}{z^2} (\Re f)_a(z) - \frac{2}{z} [(\Re f)_a]'(z). \]

It is clear that (2.5) and (2.6) are equivalent.

**Proposition 2.** A holomorphic function $F: B_n \to C$ is in $\Lambda_a(\partial B_n)$ if and only if the slice functions $F_a, a \in \partial B_n$, are uniformly bounded in $\Lambda_a(\partial B_1)$.

**Proof.** This follows from Rudin's result [3] and the proof of Theorem 1.

A comment is in order. If one considers the latter half of the proof of Theorem 1, one sees that $\Re f = F$ is much more smooth on curves $\gamma(t)$ whose tangents lie in the "complex tangential direction." However, the function $f(z) = \log(1 - z_1^2 - z_2^2)$ achieves the proper growth estimate on curves $\gamma(t) = e^{i\theta} (||w|| = 1)$ whose tangents lie in the real direction.

**References**


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