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THE ZYGMUND CONDITION FOR BLOCH FUNCTIONS IN THE BALL IN C^n

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ABSTRACT. In this paper we prove the equivalence of the Bloch condition for a holomorphic function f on the ball B_n with the Zygmund second difference condition for a suitable primitive F of f.

Introduction. If B_1 is the open unit disc in the complex plane and f is holomorphic on B_1, we say that f is a Bloch function if there exists a positive number M such that

$$|f'(z)|(1-|z|^2) < M$$

for all $z \in B_1$. When equipped with an appropriate norm the linear space of Bloch functions becomes a nonseparable Banach space. There are many equivalent conditions that a function can satisfy to be a Bloch function (see Pommerenke [2] or Cima [1]). In a recent thesis, Richard Timoney [4] has done an exhaustive study of properties of Bloch functions on domains in C^n. In particular his work includes the theory of Bloch functions on the ball

$$B_n = \left\{ z \in C^n : \|z\| = \sqrt{|z_j|^2} < 1 \right\}.$$

He has shown that all the known characterizations, save two, that are equivalent for the case of $n = 1$ are valid for $n > 1$. One of these two characterizations is the second difference condition of Zygmund [5]. We will establish the equivalence of this condition for the B_n case.

1. Preliminaries. Assume f is a holomorphic function of $B_n \to C$. For u and v vectors in C^n, $z \in B_n$ and $\langle u, v\rangle = \sum_{j=1}^n u_j \bar{v}_j$ the Bergman metric is given by

$$H_z(u, \bar{v}) = \left(\frac{n+1}{2}\right)\left[\frac{(1-\|z\|^2)\langle u, \bar{v}\rangle + \langle u, \bar{z}\rangle\langle z, \bar{v}\rangle}{(1-\|z\|^2)^2}\right].$$

For each $z \in B_n$ define

$$Q_f(z) \equiv \sup\left\{ |(\nabla_z f)(x)| / H_z(x, \bar{x})^{1/2};\ x \in C^n, x \neq 0 \right\}$$

where $(\nabla_z f)(x) = \sum_{j=1}^n (\partial f/\partial z_j)(z) x_j$.

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DEFINITION 1. *A holomorphic function f: $B_n \to C$ is called a Bloch function if* $\sup\{Q_f(z): z \in B_n\} < \infty$.

In considering this definition, a certain amount of pathology immediately enters. In fact one observes by studying the metric that if f is a Bloch function, then the growth of its directional derivative in the radial direction is $O((1 - \|z\|^2)^{-1})$, whereas the growth in directions orthogonal to the radial direction is only $O((1 - \|z\|^2)^{-1/2})$. Timoney [4] shows that the Bloch condition is equivalent to the following condition:

$$\sup\{\|\nabla_z f\|(1 - \|z\|^2); z \in B_n\} < \infty. \tag{1.1}$$

Assume F is in the ball algebra of B_1, i.e., F is continuous on $\bar{B}_1$ and holomorphic in B_1. Further assume that the boundary values of F satisfy

$$|F(e^{i(\theta+h)}) + F(e^{i(\theta-h)}) - 2F(e^{i\theta})| < A|h| \tag{1.2}$$

for all real numbers h and some positive number A, independent of θ. In [5] it was shown that a function f, holomorphic on B_1, satisfies the Bloch condition if and only if its primitive $F(z) \equiv \int_0^z f(\zeta)\, d\zeta$ is in the ball algebra of B_1 and satisfies condition (1.2).

We consider in this note C^1 curves γ mapping $\mathbf{R} \to \partial B_n$ such that $|\gamma'(t)| = 1$ for all t. We refer to these as normalized C^1 curves. With this class of curves in mind we make the following definition.

DEFINITION 1.2. *Let F be in the ball algebra of B_n. We say that F satisfies condition $\Lambda_*(\partial B_n)$ if there exists a positive number A such that for all normalized C^1 curves γ in B_n,*

$$|F(\gamma(t + h)) + F(\gamma(t - h)) - 2F(\gamma(t))| < A|h|$$

for all t and h in $\mathbf{R}$.

Finally, if f is holomorphic on B_n with expansion in terms of homogeneous polynomials given by $f(z) = \sum_{k=0}^{\infty} F_k(z)$ define the radial derivative $\mathcal{R}f$ of f by the formula

$$\mathcal{R}f(z) \equiv \sum_{k=1}^{\infty} kF_k(z).$$

2. The principal result. If we are given a function f holomorphic on B_n, define a function

$$F(z) = (\mathcal{I}f)(z) \equiv \int_0^1 f(tz)\, dt.$$

One checks that $\mathcal{R}(\mathcal{I}f)(z) = \mathcal{R}F(z) = f(z) - F(z)$.

THEOREM 1. *A function f holomorphic on B_n is in the Bloch space of B_n if and only if $\mathcal{I}f$ satisfies the $\Lambda_*(\partial B_n)$ condition.*

PROOF. Assume first that f is a Bloch function. For $a \in \partial B_n$ the slice functions are defined by $f_a(\lambda) = f(\lambda a)$, $\lambda \in B_1$. Since

$$(\mathcal{I}f)_a(\lambda) \equiv F_a(\lambda) = \frac{1}{\lambda} \int_0^\lambda f_a(\zeta)\, d\zeta \tag{2.1}$$

we see that each F_a is in $\Lambda_*(\partial B_1)$. Also each member of the family $\{F_a;\, a \in \partial B_n\}$ has its oscillation $\omega(F_a, \delta) = O(\delta \log \delta)$, uniformly in a. F can be extended to a function on $\bar{B}_n$ by using the values on the slices. Now with $b \in \partial B_n$, $0 < r < 1$,

$$|F(a) - F(b)| < M(1 - r)\log(1 - r) + |F_a(r) - F_b(r)|.$$

This shows that F is continuous at a and hence is in the ball algebra.

Now fix a normalized C^1 curve γ with range in ∂B_n and let $h > 0$ be given. If $g(t)$ is any function defined on $\mathbf{R}$ set $Ag(t) = g(t + h) - g(t)$. With $r = 1 - h$ we write

$$F(\gamma(t)) = (F(\gamma(t)) - F(r\gamma(t))) + (F(r\gamma(t)))$$

and show that A^2 of each expression in parentheses is $O(h)$, independent of t. Since $|f(z)| = O(\log(1 - |z|))$ one easily verifies that

$$F(\gamma(t)) - F(r\gamma(t)) = (1 - r)f(r\gamma(t)) + \int_r^1 (1 - s)\, \nabla f(s\gamma(\theta)) \circ \gamma(\theta)\, ds.$$

The integral in this equality is $O(h)$. Also

$$Af(r\gamma(t)) = \int_0^h \nabla f(\zeta) \circ \zeta' = O(1)$$

where $\zeta(p) = r\gamma(t + p)$. Hence

$$A^2[F(\gamma(t)) - F(r\gamma(t))] = O(h)$$

uniformly in t. The expression $A^2F(r\gamma(t))$ involves three terms:

$$r\int_0^h (\gamma'(t + p) - \gamma'(t)) \circ \nabla F(r\gamma(t + p))\, dp, \tag{2.2}$$

$$r\int_0^h \gamma'(t) \circ (\nabla F(r\gamma(t + p)) - \nabla F(r\gamma(t - p)))\, dp, \tag{2.3}$$

$$r\int_0^h (\gamma'(t) - \gamma'(t - p)) \circ \nabla F(r(\gamma(t - p)))\, dp. \tag{2.4}$$

By the definition

$$\left|\frac{\partial F}{\partial z_j}(r\gamma(t))\right| = \left|r\int_0^1 \nabla f(ru\gamma(t)) \circ \gamma'(t)u\, du\right| < M \cdot |\log h|.$$

Thus, expressions (2.2) and (2.4) are $O(1)$ uniformly in t. Similarly

$$\|\nabla F(r\gamma(t)) - \nabla F(r\gamma(s))\| = O(\log(1 - r))$$

and hence (2.3) is $O(1)$.

For the converse we observe that for each $0 < \alpha < 1$, the space $\Lambda_\alpha(B_1)$ of functions in the ball algebra with boundary values in the Lip α space contains $\Lambda_*(B_1)$. Further, bounded subsets of Λ_* are bounded in Λ_α. Fix $z = re_1$ in B_n with $\|e_1\| = 1$, $0 < r < 1$, and let $\{e_j\}_{j=1}^n$ be an orthonormal basis for C^n. Let D_j be the derivative in the e_j direction. We can apply a result of Rudin [3] to draw the following conclusions. Since $\{F_w\}$ is a norm-bounded subset of $\Lambda_{1/2}(B_1)$,

$$(\Re F)(z) = O\big((1 - \|z\|)^{-1/2}\big)$$

and

$$D_jf(z) = O\big((1 - \|z\|)^{-1}\big), \qquad 2 < j < n.$$

Since $F_{e_1}(r) + (\Re F)_{e_1}(r) = f_{e_1}(r)$ we apply the one variable result to conclude

$$D_1 f(z) = Df_{e_1}(r) = O\left((1 - \|z\|)^{-1}\right).$$

The estimates are uniform.

The referee has pointed out that our proof yields the following equivalences.

PROPOSITION 1. *A homolomorphic function f: $B_n \to C$ is a Bloch function if and only if the slice functions $F_a = (\mathcal{P}f)_a$, $a \in \partial B_n$, are uniformly bounded in $\Lambda_*(\partial B_1)$.*

PROOF. From [4] a function f: $B_n \to C$ is a Bloch function if and only if

$$\sup_{z \in B_n} |(\Re f)(z)|(1 - \|z\|^2) < \infty. \tag{2.5}$$

The functions $(\mathcal{P}f)_a$, $a \in \partial B_n$, are uniformly bounded in $\Lambda_*(\partial B_1)$ if and only if

$$\sup_{|z|<1,\, a \in \partial B_n} |[(\mathcal{P}f)_a]''(z)|(1 - |z|^2) < \infty. \tag{2.6}$$

A computation with (2.1) shows that

$$\begin{aligned}[(\mathcal{P}f)_a]''(z) &= \frac{1}{z^2}(\Re f)_a(z) - \frac{2}{z^2}[f_a(z) - (\mathcal{P}f)_a(z)] \\ &= \frac{1}{z^2}(\Re f)_a(z) - \frac{2}{z}[(\mathcal{P}f)_a]'(z).\end{aligned}$$

It is clear that (2.5) and (2.6) are equivalent.

PROPOSITION 2. *A holomorphic function F: $B_n \to C$ is in $\Lambda_*(\partial B_n)$ if and only if the slice functions F_a, $a \in \partial B_n$, are uniformly bounded in $\Lambda_*(\partial B_1)$.*

PROOF. This follows from Rudin's result [3] and the proof of Theorem 1.

A comment is in order. If one considers the latter half of the proof of Theorem 1, one sees that $\mathcal{P}f = F$ is much more smooth on curves $\gamma(t)$ whose tangents lie in the "complex tangential direction." However, the function $f(z) = \log(1 - z_1^2 - z_2^2)$ achieves the proper growth estimate on curves $\gamma(t) = e^{it}w$ ($\|w\| = 1$) whose tangents lie in the real direction.

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