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A REMARK ON COMPLEMENTED SUBSPACES OF UNITARY MATRIX SPACES

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ABSTRACT. THEOREM A. *Let P be a bounded projection in a unitary matrix space C_E. Then either PC_E or $(I - P)C_E$ contains a subspace which is isomorphic to C_E and complemented in C_E.*

1. Introduction. Let E be a symmetric sequence space, i.e. a Banach space of sequences so that the standard unit vectors $\{e_n\}_{n=1}^{\infty}$ (defined by $e_n(i) = \delta_{n,i}$) form a 1-symmetric basis of E. We denote by C_E the Banach space of all compact operators x on l_2 so that the sequence $s(x) = (s_n(x))_{n=1}^{\infty}$ of s-numbers of x (i.e. the eigenvalues of $(x^*x)^{\frac{1}{2}}$) belongs to E, normed by $\|x\|_{C_E} = \|s(x)\|_E$. The spaces C_E are called unitary matrix spaces. For their study see [3] and [5].

The main result of the present paper is Theorem A, stated in the abstract. It may be used in proving that certain unitary matrix spaces (the spaces C_p, $1 < p < \infty$, for example) are primary. The problem of whether every unitary matrix space is primary is however still open. Theorem A also has a local version which is discussed at the end of the paper.

We use standard terminology from Banach space theory, see [6]. Also we identify operators x on l_2 with their matrices $(x(i,j))$ with respect to some fixed orthonormal basis in l_2. The standard unit matrices $\{e_{n,k}\}_{n,k=1}^{\infty}$ are defined by $e_{n,k}(i,j) = \delta_{n,i} \cdot \delta_{k,j}$. If $\{i_k\}_{k=1}^{\infty}$ and $\{j_k\}_{k=1}^{\infty}$ are increasing sequences of positive integers, then $Q(\{i_k\}, \{j_k\})$ is the projection defined by

$$(Q(\{i_k\}, \{j_k\})x)(i,j) = \begin{cases} x(i,j) & \text{if } i = i_k \text{ and } j = j_l \text{ for some } k \text{ and } l, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Clearly, this projection has norm one on every unitary matrix space.

Another important projection is the *triangular projection* T defined by

$$(Tx)(i,j) = \begin{cases} x(i,j) & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

For every symmetric sequence space E let us denote

$$T_E = \{x \in C_E;\ x(i,j) = 0 \text{ if } i > j\}. \tag{1.3}$$

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PROPOSITION 1.1. *Let E be a symmetric sequence space. Then the following are equivalent*

(i) *T is bounded in C_E;*

(ii) *for every $\lambda \neq 1$ the operator $V_\lambda = \lambda T + I - T$ is bounded in C_E;*

(iii) *T_E is isomorphic to C_E.*

PROOF. The eigenvalence (i) $\Leftrightarrow$ (ii) follows from the formula

$$T = (I - V_\lambda)/(1 - \lambda).$$

The eigenvalence (i) $\Leftrightarrow$ (iii) is proved in [2]. □

A *triangle* is a double sequence of the form $\{x_{i,j}\}_{1 \leq i \leq j < \infty}$. A *subtriangle* of $\{x_{i,j}\}_{i \leq j}$ is a triangle of the form $\{x_{i_k,j_l}\}_{k \leq l}$ where $\{i_k\}$ and $\{j_l\}$ are increasing sequences with $i_k \leq j_k$ for every k. If we consider a triangle $\{x_{i,j}\}_{i \leq j}$ of elements of a Banach space as a basic sequence, we shall always assume that it is a basic sequence in the lexicographic ordering:

$$x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}, \dots. \tag{1.4}$$

DEFINITION 1.2. Let $\Delta = \{\alpha_{i,j}\}_{i \leq j}$ be a triangle of numbers. We denote by $L(\Delta)$ the set of all numbers α so that for some subtriangle $\{\alpha_{i_k,j_l}\}_{k \leq l}$ the following limits exist

$$\alpha_k = \lim_{l \to \infty} \alpha_{i_k,j_l}, \qquad \alpha = \lim_{k \to \infty} \alpha_k. \tag{1.5}$$

Note that if Δ is bounded, i.e. $\sup_{i,j}|\alpha_{i,j}| < \infty$, then $L(\Delta) \neq \emptyset$. This can be proved by standard compactness arguments and a diagonal process. Note also that if $\alpha \in L(\Delta)$, then the subtriangle $\{\alpha_{i_k,j_l}\}_{k \leq l}$ can be chosen so that the convergence in (1.5) is arbitrarily fast.

2. The main lemma. The following lemma is our main tool.

LEMMA 2.1. *Let E be any symmetric sequence space, and let S: $T_E \to C_E$ be a nonzero bounded operator. Let $x_{i,j} = Se_{i,j}$, $\alpha_{i,j} = x_{i,j}(i,j)$, $1 \leq i \leq j < \infty$, and let $\alpha \in L(\{\alpha_{i,j}\}_{i \leq j})$. Then for every $0 < \varepsilon < 1$ there exist increasing sequences of positive integers $\{m_\nu\}_{\nu=1}^\infty$ and $\{n_\nu\}_{\nu=1}^\infty$, satisfying $m_\nu \leq n_\nu < m_{\nu+1}$ for every ν, so that if $Q = Q(\{m_\nu\}, \{n_\nu\})$ is defined by* (1.1) *and if we define $U = (QS - \alpha I)TQ$: $C_E \to C_E$, then $\|U\| \leq \varepsilon$.*

PROOF. By passing to a subtriangle if necessary, we may assume that for some numbers $\{\alpha_i\}_{i=1}^\infty$, we have

$$|\alpha_{i,j} - \alpha_i| \leq \varepsilon \cdot 8^{-i-j}, \qquad |\alpha_i - \alpha| \leq \varepsilon \cdot 8^{-i}. \tag{2.1}$$

Now, for each fixed i, $\{e_{i,j}\}_{j=i}^\infty$ is isometrically equivalent to the unit vector basis of l_2, so $x_{i,j} = Se_{i,j} \to 0$ weakly as $j \to \infty$. By standard perturbation arguments we can assume that for some increasing sequences of positive integers $\{\tau_k\}_{k=1}^\infty$, $\{i_k\}_{k=1}^\infty$ and $\{j_k\}_{k=1}^\infty$ with

$$\tau_k < i_k \leq j_k \leq \tau_{k+1}, \qquad k = 1, 2, \dots, \tag{2.2}$$

we have for every $k \leq l$,

$$x_{i_k,j_l}(i,j) = 0 \quad \text{if either } \max\{i,j\} \leq \tau_l \text{ or } \max\{i,j\} > \tau_{l+1}. \tag{2.3}$$

For every n, k, l with $k < l$ we define

$$\lambda(n, k, l) = x_{i_k j_l}(i_n, j_l). \tag{2.4}$$

Note that $\lambda(k, k, l) = \alpha_{i_k j_l}$, and that in general $|\lambda(n, k, l)| \leq \|x_{i_k j_l}\| \leq \|S\|$.

We construct now an increasing sequence $\{l_\nu\}_{\nu=1}^\infty$ of positive integers and numbers $\lambda(n, k)$ so that for every $n, k \leq \nu$,

$$|\lambda(n, k, l_\nu) - \lambda(n, k)| \leq \varepsilon \cdot 8^{-n-k-\nu}. \tag{2.5}$$

Indeed, every subsequence of $\{\lambda(n, k, l)\}_{l=\max\{n,k\}}^\infty$ has a further convergent subsequence. Let $\{l_i^{(1)}\}_{i=1}^\infty$ be an increasing sequence, and let $\lambda(1, 1)$ be such that $|\lambda(1, 1, l_i^{(1)}) - \lambda(1, 1)| \leq \varepsilon \cdot 8^{-2-i}$ for every i. If $\{l_i^{(m)}\}_{i=m}^\infty$ and $\{\lambda(n, k)\}_{\max\{n,k\}=m}$ have been defined, let $\{\lambda(n, k)\}_{\max\{n,k\}=m+1}$ be numbers so that for some subsequence $\{l_i^{(m+1)}\}_{i=m+1}^\infty$ of $\{l_i^{(m)}\}_{i=m+1}^\infty$ we have

$$|\lambda(n, k, l_i^{(m+1)}) - \lambda(n, k)| \leq \varepsilon \cdot 8^{-n-k-i} \tag{2.6}$$

for every i and every n, k with $\max\{n, k\} = m + 1$. By defining $l_\nu = l_\nu^{(\nu)}$, we clearly get that (2.5) holds for every $n, k \leq \nu$.

Note that by (2.1),

$$\lambda(k, k) = \alpha_{i_k} \quad \text{for every } k. \tag{2.7}$$

Let n_0 and k be given and let $\nu = \max\{n_0, k\}$. Then by (2.5),

$$\left(\sum_{n=1}^{n_0} |\lambda(n, k)|^2\right)^{1/2} \leq \left(\sum_{n=1}^{n_0} |\lambda(n, k, l_\nu)|^2\right)^{1/2} + \varepsilon \leq \|x_{i_k j_{l_\nu}}\| + \varepsilon \leq \|S\| + \varepsilon. \tag{2.8}$$

It follows that for some increasing sequence $\{k_\mu\}_{\mu=1}^\infty$ of positive integers and for some subsequence of $\{l_\nu\}_{\nu=1}^\infty$ which we continue to denote by $\{l_\nu\}_{\nu=1}^\infty$ for convenience, we have $k_\mu \leq l_\mu < k_{\mu+1}$ for every μ, and

$$|\lambda(k_\mu, k_\nu)| \leq \varepsilon \cdot 8^{-\mu-\nu} \quad \text{for every } \nu < \mu. \tag{2.9}$$

We now claim that

$$\lambda(k_\mu, k_\sigma) \to 0 \quad \text{as } \sigma \to \infty, \text{ for every } \mu. \tag{2.10}$$

Indeed, if (2.10) is false for some μ, then for some $a \neq 0$ and some increasing sequence $\{\sigma_t\}_{t=1}^\infty$, we have $|\lambda(k_\mu, k_{\sigma_t}) - a| \leq 2^{-t}$ for every t. Let N be such that $\|S\|(N + 1)^{1/2} < (N + 1)|a| - 1 - \varepsilon$, and choose ν so that $\nu \geq \max\{k_\mu, k_{\sigma_{2N}}\}$. Then, by (2.5),

$$\begin{aligned}\|S\|(N + 1)^{1/2} &\geq \left\|\sum_{t=N}^{2N} x_{i_{k_{\sigma_t}} j_{l_\nu}}\right\| \geq \left|\sum_{t=N}^{2N} \lambda(k_\mu, k_{\sigma_t}, l_\nu)\right| \\ &\geq \left|\sum_{t=N}^{2N} \lambda(k_\mu, k_{\sigma_t})\right| - \sum_{t=N}^{2N} \varepsilon \cdot 8^{-\mu-\sigma_t-\nu} \\ &\geq (N + 1)|a| - \sum_{t=N}^{2N} 2^{-t} - \varepsilon \geq (N + 1)|a| - 1 - \varepsilon. \end{aligned} \tag{2.11}$$

This contradicts the choice of N and thus proves (2.10).

By passing to further subsequences of $\{k_\mu\}_{\mu=1}^\infty$ and $\{l_\nu\}_{\nu=1}^\infty$ if necessary, we may assume that

$$|\lambda(k_\mu, k_\nu)| < \varepsilon \cdot 8^{-\mu-\nu}, \qquad \mu \neq \nu, \tag{2.12}$$

and that

$$k_\mu < l_\mu < k_{\mu+1}, \qquad \mu = 1, 2, \ldots . \tag{2.13}$$

Let for $\mu, \nu = 1, 2, \ldots$

$$m_\mu = i_{k_{2\mu-1}}, \qquad n_\nu = j_{l_{2\nu}}, \tag{2.14}$$

and let

$$Q = Q(\{m_\nu\}, \{n_\nu\}) \tag{2.15}$$

be the projection defined by (1.1). Note that by (2.2), (2.3), (2.13), (2.14) and (2.15), we have

$$Qx_{m_\mu, n_\nu} \in T_E, \qquad \mu < \nu. \tag{2.16}$$

Define

$$U = (QS = \alpha I)TQ, \tag{2.17}$$

where T is the triangular projection (1.2).

Let $\sum_{i,j=1}^\infty t_{i,j} e_{i,j}$ be a normalized element of C_E, so that $t_{i,j} \neq 0$ only for finitely many pairs (i, j). Then using (in this order) (2.13), (2.14), (2.16), the fact that for fixed i, $\{e_{i,j}\}_{j=1}^\infty$ are isometrically equivalent to the unit vector basis of l_2, (2.5), (2.1) and (2.12), we get

$$\begin{aligned}
\left\| U \sum_{i,j=1}^\infty t_{i,j} e_{i,j} \right\| &= \left\| \sum_{\mu<\nu} t_{m_\mu, n_\nu} \left(Qx_{m_\mu, n_\nu} - \alpha e_{m_\mu, n_\nu} \right) \right\| \\
&< \left\| \sum_{\mu<\nu} t_{m_\mu, n_\nu} \left(\lambda(k_{2\mu-1}, k_{2\mu-1}, l_{2\nu}) - \alpha \right) e_{m_\mu, n_\nu} \right\| \\
&\quad + \left\| \sum_{\mu<\nu} t_{m_\mu, n_\nu} \sum_{\substack{\sigma=1 \\ \sigma\neq\mu}}^{\nu} \lambda(k_{2\sigma-1}, k_{2\mu-1}, l_{2\nu}) e_{m_\mu, n_\nu} \right\| \\
&< \sum_{\mu=1}^\infty \left(\sum_{\nu=1}^\infty |t_{m_\mu, n_\nu}|^2 \cdot |\lambda(k_{2\mu-1}, k_{2\mu-1}, l_{2\nu}) - \alpha|^2 \right)^{1/2} \\
&\quad + \sum_{\substack{\mu,\sigma=1 \\ \mu\neq\sigma}}^\infty \left(\sum_{\nu=\max\{\mu,\sigma\}}^\infty |t_{m_\mu, n_\nu}|^2 \cdot |\lambda(k_{2\sigma-1}, k_{2\mu-1}, l_{2\nu})|^2 \right)^{1/2} \\
&< 2\varepsilon \sum_{\mu=1}^\infty 8^{-\mu} + 2\varepsilon \sum_{\mu,\sigma=1}^\infty 8^{-\mu-\sigma} < 2\varepsilon\left(\frac{1}{7} + \frac{1}{49} \right) < \varepsilon.
\end{aligned} \tag{2.18}$$

So, $\|U\| < \varepsilon$ and the Lemma is proved. □

COROLLARY 2.2. *Let E, S: $T_E \to C_E$ and $\alpha_{i,j} = (Se_{i,j})(i,j)$, $1 \le i \le j < \infty$, be as in the statement of Lemma* 2.1, *and assume that $0 \neq \alpha \in L(\{\alpha_{i,j}\}_{i \le j})$. Then there exists a subtriangle $\{y_{\mu,\nu}\}_{\mu \le \nu}$ of $\{Se_{i,j}\}_{i \le j}$ which is $2\|S\|\,|\alpha|^{-1}$-equivalent to $\{e_{\mu,\nu}\}_{\mu \le \nu}$. If, moreover, $S(T_E) \subset T_E$, then $\{y_{\mu,\nu}\}_{\mu \le \nu}$ can be chosen so that $[y_{\mu,\nu}]_{\mu \le \nu}$ is $2\|S\|\,|\alpha|^{-1}$-complemented in T_E.*

PROOF. Let $0 < \varepsilon < \min\{|\alpha|, \|S\|\}/20$. By Lemma 2.1, choose increasing sequences $\{m_\mu\}$ and $\{n_\nu\}$ of positive integers with $m_\nu \le n_\nu < m_{\nu+1}$ for every ν, and so that if we define $Q = Q(\{m_\nu\}, \{n_\nu\})$ and $U = (QS - \alpha I)TQ$, then $\|U\| \le \varepsilon$. Let $y_{\mu,\nu} = Se_{m_\mu,n_\nu}$, $\mu \le \nu$. Then for every $x = \sum_{\mu \le \nu} t_{\mu,\nu} e_{m_\mu,n_\nu} \in C_E$, we have

$$\Big\| \sum_{\mu \le \nu} t_{\mu,\nu} y_{\mu,\nu} \Big\| = \|Sx\| \ge \|QSx\| \ge |\alpha| \cdot \|x\| - \|QSx - \alpha x\|$$

$$\ge (|\alpha| - \|U\|)\|x\| \ge (|\alpha| - \varepsilon)\|x\| \ge |\alpha| \cdot \|x\|/2. \tag{2.19}$$

Since $\|\sum_{\mu \le \nu} t_{\mu,\nu} y_{\mu,\nu}\| = \|Sx\| \le \|S\| \cdot \|x\|$, we get that $\{y_{\mu,\nu}\}_{\mu \le \nu}$ is $2\|S\|\,|\alpha|^{-1}$-equivalent to $\{e_{m_\mu,n_\nu}\}_{\mu \le \nu}$. This proves the first assertion since $\{e_{m_\mu,n_\nu}\}_{\mu \le \nu}$ is isometrically equivalent to $\{e_{\mu,\nu}\}_{\mu \le \nu}$.

Now assume that $ST_E \subset T_E$. Let $Y = [y_{\mu,\nu}]_{\mu \le \nu} = SQT_E$, and define

$$P_0 = \alpha^{-1}(SQ - U)_{|T_E}, \tag{2.20}$$

$$V = I + \alpha^{-1}U. \tag{2.21}$$

Clearly, P_0 is a projection in T_E and $\|P_0\| \le 21\|S\|/20|\alpha|$. Also, $\|V - I\| \le |\alpha|^{-1}\varepsilon \le 1/20$, and so V is an automorphism of T_E. It follows that

$$P = VP_0V^{-1} \tag{2.22}$$

is a projection in T_E, $\|P\| \le \|V\| \cdot \|P_0\| \cdot \|V^{-1}\| \le 2\|S\|\,|\alpha|^{-1}$, and since $VP_0 = \alpha^{-1}SQ_{|T_E}$, the range of P is exactly Y. □

3. Proof of Theorem A. We are now ready to prove Theorem A stated in the abstract. We actually prove a somewhat stronger result, namely:

THEOREM 3.1. *Let E be a symmetric sequence space, let X be either C_E or T_E and let P be any bounded projection in X. Then either PX or $(I - P)X$ contains a subspace which is isomorphic to X and complemented in X.*

PROOF OF THEOREM 3.1 FOR $X = T_E$. Let us denote for $1 \le i \le j < \infty$, $a_{i,j} = Pe_{i,j}$, $b_{i,j} = (I - P)e_{i,j}$, $\alpha_{i,j} = a_{i,j}(i,j)$ and $\beta_{i,j} = b_{i,j}(i,j)$. Since $a_{i,j} + b_{i,j} = e_{i,j}$ for every $i \le j$, we have $\alpha_{i,j} + \beta_{i,j} = 1$, and thus either $|\alpha_{i,j}| \ge 1/2$ or $|\beta_{i,j}| \ge 1/2$ (or both). By Ramsey's theorem (see [4]) there exists an increasing sequence of positive integers $\{i_k\}_{k=1}^{\infty}$ so that either

$$|\alpha_{i_k,i_l}| \ge 1/2 \quad \text{for every } k < l, \tag{3.1}$$

or

$$|\beta_{i_k,i_l}| \ge 1/2 \quad \text{for every } k < l. \tag{3.2}$$

Without loss of generality we assume that (3.1) holds (since, otherwise (3.2) holds, and the proof is the same provided we replace P by $I - P$ and $a_{i,j}$ by $b_{i,j}$).

Choose any $\alpha \in L(\{\alpha_{i_k,i_l}\}_{k<l})$, and note that (3.1) implies that $|\alpha| \geq 1/2$. Using Lemma 2.1 and Corollary 2.2 with $S = P$, we get a subtriangle $\{y_{\mu,\nu}\}_{\mu<\nu}$ of $\{a_{i_k,i_l}\}_{k<l}$ which is $4\|P\|$-equivalent to $\{e_{\mu,\nu}\}_{\mu<\nu}$, and so that $[y_{\mu,\nu}]_{\mu<\nu} = Y$ is $4\|P\|$-complemented in T_E. Since $Y \subset PT_E$, this completes the proof in this case.

Proof of Theorem 3.1 for $X = C_E$. By the case $X = T_E$ treated above, it is enough to consider here only the spaces C_E so that $C_E \not\approx T_E$. That is, by Proposition 1.1, the spaces C_E, so that

$$V_\lambda = \lambda T + I - T \quad \text{is bounded in } C_E \text{ only for } \lambda = 1. \tag{3.3}$$

Let us denote, for every $1 \leq i, j < \infty$, $a_{i,j} = Pe_{i,j}$, $b_{i,j} = (I - P)e_{i,j}$, $\alpha_{i,j} = a_{i,j}(i,j)$ and $\beta_{i,j} = b_{i,j}(i,j)$. Using Ramsey's theorem once again, we get an increasing sequence $\{i_k\}_{k=1}^\infty$ so that either

$$|\alpha_{i_k,i_l}| \geq 1/2 \quad \text{for every } l < k, \tag{3.4}$$

or

$$|\beta_{i_k,i_l}| \geq 1/2 \quad \text{for every } l < k. \tag{3.5}$$

Again, we assume without loss of generality that (3.4) holds. By passing to a subsequence of $\{i_k\}_{k=1}^\infty$ if necessary we may assume also that for some α with $1/2 \leq |\alpha| \leq \|P\|$ we have $L(\{\alpha_{i_k,i_l}\}_{l<k}) = \{\alpha\}$. Choose some $\gamma \in L(\{\alpha_{i_k,i_l}\}_{k<l})$ and let

$$0 < \varepsilon < \|P\|/20(\|P\| + 1). \tag{3.6}$$

Applying Lemma 2.1 to $S_1 = P_{|T_E}$ and T_E we construct subsequences $\{n_\nu^{(1)}\}_{\nu=1}^\infty$ and $\{m_\nu^{(1)}\}_{\nu=1}^\infty$ of $\{i_k\}_{k=1}^\infty$ so that $m_\nu^{(1)} \leq n_\nu^{(1)} < m_{\nu+1}^{(1)}$ for every ν and so that if $Q_1 = Q(\{m_\nu^{(1)}\}, \{n_\nu^{(1)}\})$, then

$$\|(Q_1S_1 - \gamma I)TQ_1\| \leq \varepsilon. \tag{3.7}$$

Clearly, $L(\{\alpha_{m_\mu^{(1)},n_\nu^{(1)}}\}_{\nu<\mu}) = \{\alpha\}$. Now T_E is isometric in a natural way to $\tilde{T}_E = [e_{i,j}]_{j<i}$ so by applying Lemma 2.1 to $S = P_{|\tilde{T}_E}$ and to $\tilde{T}_E$ and $I - T$ instead of T_E and T respectively, we get subsequences $\{n_\nu\}_{\nu=1}^\infty$ and $\{m_\nu\}_{\nu=1}^\infty$ of $\{n_\nu^{(1)}\}_{\nu=1}^\infty$ and $\{m_\nu^{(1)}\}_{\nu=1}^\infty$ respectively, with $m_\nu \leq n_\nu < m_{\nu+1}$ for every ν, so that if $Q = Q(\{m_\nu\}, \{n_\nu\})$, then

$$\|(QS - \alpha I)(I - T)Q\| \leq \varepsilon. \tag{3.8}$$

Since the family $\{Q, Q_1, T\}$ is commutative, it is clear that (3.7) holds also with Q instead of Q_1. Let

$$W = V_{(\gamma/\alpha)} \cdot Q = (\gamma T/\alpha + I - T)Q. \tag{3.9}$$

Then for every $x \in C_E$ with only finitely many nonzero coordinates,

$$\begin{aligned} \|(\alpha^{-1}QPQ - W)x\| &\leq \|(QS_1 - \gamma I)TQx\|/|\alpha| + \|(QS - \alpha I)(I - T)Qx\|/|\alpha| \\ &\leq 2\varepsilon\|x\|/|\alpha|. \end{aligned} \tag{3.10}$$

In particular this implies that W is bounded in C_E. Let J be the isometry of C_E onto QC_E defined by $Je_{\mu,\nu} = e_{m_\mu,n_\nu}$, then $J^{-1}WJ = V_{(\gamma/\alpha)}$. By the assumption (3.3) we get that $\alpha = \gamma$ and thus $W = Q$.

Let $U = \alpha^{-1}PJ$. Then for every $x \in C_E$,

$$\|Ux\| \geq \|QUx\| \geq \|Jx\| - \|\alpha^{-1}QPQJx - QJx\| \geq (1 - 2\varepsilon|\alpha|^{-1})\|x\| \geq \frac{4}{5}\|x\|. \tag{3.11}$$

It follows that U is an isomorphism (with constant $\leq 20\|P\|/9$) from C_E onto some subspace Z of PC_E. Let

$$R_0 = \alpha^{-1}PQ + Q - \alpha^{-1}QPQ, \tag{3.12}$$

$$V = I + \alpha^{-1}QPQ - Q. \tag{3.13}$$

Then R_0 is a bounded projection in C_E, and by (3.10), $\|V - I\| \leq 1/5$. So, V is an automorphism of C_E and $R = VR_0V^{-1}$ is a projection in C_E with $\|R\| \leq 4\|P\|$. Since $VR_0 = \alpha^{-1}PQ$, the range of R is Z. This completes the proof of Theorem 3.1. □

4. Applications. Recall that a Banach space X is *primary* if for every bounded projection P on X, either $PX \approx X$ or $(I - P)X \approx X$. For $1 \leq p < \infty$, let $C_p = C_{l_p}$, and let $C_\infty = C_{c_0}$. It is known that for $1 < p < \infty$, C_p is primary (see [1]). We strengthen this somewhat as follows.

COROLLARY 4.1. *Let C_E be a unitary matrix space so that for some $1 \leq p \leq \infty$,*

$$C_E \approx (C_E \oplus C_E \oplus \cdots \oplus C_E \oplus \cdots)_{l_p} \tag{4.1}$$

(*where if $p = \infty$, the direct sum is taken in the sense of c_0*). *Then C_E is primary. In particular, the spaces C_p, $1 \leq p \leq \infty$, are primary.*

The corollary follows easily from Theorem 3.1, the decomposition method (see [6, p. 54]) and the fact (see [1]) that for every $1 \leq p \leq \infty$,

$$C_p \approx (C_p \oplus C_p \oplus \cdots \oplus C_p \oplus \cdots)_{l_p}.$$

These arguments show that Corollary 4.1 holds also for T_E instead of C_E.

PROBLEM 4.2. *Is every unitary matrix space a primary Banach space*?

Let us pass to a *local* (i.e. finite dimensional) *version of Theorem* 3.1. If E is a symmetric sequence space, then C_E^n denotes the space of all $n \times n$ matrices with the norm induced from C_E. Let $\{A_n\}_{n=1}^{\infty}$ be pairwise disjoint subsets of the positive integers so that A_n has exactly n elements, and let

$$S_E = \left\{ x \in C_E;\ x(i,j) = 0 \text{ if } (i,j) \notin \bigcup_{n=1}^{\infty} A_n \times A_n \right\}. \tag{4.2}$$

S_E can be considered in the obvious way as the direct sum $S_E = (\sum_{n=1}^{\infty} \oplus C_E^n)_E$. The importance of S_E stems from the easily proved fact that C_E is finitely represented in S_E.

Using the same ideas as in the proofs of Lemma 2.1 and Theorem 3.1, we are able to prove the following results.

COROLLARY 4.2. *Let k be a positive integer and let $1 \leq M < \infty$. Then there exists a positive integer $n = n(k, M)$ so that if E is any symmetric sequence space and if P is any projection in C_E^n with $\|P\| \leq M$, then either PC_E^n or $(I - P)C_E^n$ contains a subspace which is $4(M + 1)$-isomorphic to C_E^k and $4(M + 1)$-complemented in C_E^n.*

COROLLARY 4.3. *Let E be any symmetric sequence space and let P be a bounded projection in S_E. Then either PS_E or $(I - P)S_E$ contains a subspace which is isomorphic to S_E and complemented in S_E.*

COROLLARY 4.4. *For $1 \leq p \leq \infty$, the space $S_p = (\sum_{n=1}^{\infty} \oplus C_p^n)_{l_p}$ is primary.*

Here, as before, if $p = \infty$ the direct sum is taken in the sense of c_0.

In proving Corollary 4.2 we use Ramsey's theorem for finite sets many times. The estimate on the growth of n as a function of k we obtain is therefore very bad. In proving Corollary 4.3, we also use the fact (easily proved by the decomposition method) that if $\{n_k\}_{k=1}^{\infty}$ is any sequence of positive integers so that $\sup_k n_k = \infty$, then $S_E \approx (\sum_{k=1}^{\infty} \oplus C_E^{n_k})_E$.

Problem 4.5. *Let E be a symmetric sequence space. Is S_E a primary Banach space?*

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