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THE EQUALITY OF UNILATERAL DERIVATES

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ABSTRACT. C. J. Neugebauer has shown that if f is a continuous function of bounded variation defined on the real line, then the set E where the upper right derivate differs from the upper left derivate is of measure zero and first category. Here it is shown that this result is best possible; that is, given any measure zero first category set K, there is a continuous function of bounded variation for which $K \subsetneq E$. It is also shown that if f is monotone, then E is σ-porous. This result can be used to provide an elementary proof of the fact that for an arbitrary function f the left and right essential cluster sets are identical except at a σ-porous set of points, a result first proved by L. Zajíček.

1. Introduction. Let f be a real valued function defined on the real line R and let

$$E = \{x: f^-(x) \neq f^+(x) \text{ or } f_-(x) \neq f_+(x)\},$$

where $f^-(x)$ $[f^+(x)]$ and $f_-(x)$ $[f_+(x)]$ denote the left [right] upper and lower derivates of f at x, respectively. C. J. Neugebauer [5] showed that if f is of bounded variation and continuous then E is of measure zero and first category. Using this result he provided an elementary proof of a result of M. Kulbacka [3] which states that for an arbitrary $f: R \to R$ the left and right essential cluster sets are equal except at those points in a set which is of measure zero and first category; in symbols, the set

$$B = \{x: C_e^-(f, x) \neq C_e^+(f, x)\}$$

is of measure zero and first category. L. Zajíček [6] has subsequently refined Kulbacka's result by showing that B must be a σ-porous set.

The notion of a σ-porous set was introduced by E. P. Dolženko [2]. If S is a set in R, the *porosity of S at the point x* in R is defined to be

$$\limsup_{r \downarrow 0} l(x, r, S)/r,$$

where $l(x, r, S)$ denotes the length of the largest open interval contained in $(x - r, x + r) \cap (R \setminus S)$. The set S is called *porous* if it has positive porosity at each of its points, and it is called *σ-porous* if it is a countable union of porous sets. Clearly, σ-porous sets must be of measure zero and first category. On the other hand, Zajíček [7] has constructed perfect measure zero sets which are not σ-porous.

A natural line of thought is to see if B can be shown to be σ-porous by first obtaining an appropriate theorem concerning the derivates of some class of real functions and then applying Neugebauer's argument. This is accomplished in §2 of

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the present paper with the class being the collection of monotone functions. In §3 we show that the class of continuous bounded variation functions is not small enough for our purpose in that given any measure zero first category set K, there is a continuous bounded variation function such that $K \subseteq E$.

2. Derivates of monotone functions. We begin by considering a theorem concerning the class of monotone functions and examine some consequences.

THEOREM 1. *If f: $R \to R$ is monotone, then the set $E = \{x\colon f^-(x) \neq f^+(x)$ or $f_-(x) \neq f_+(x)\}$ is σ-porous.*

PROOF. Suppose that f is nondecreasing. We shall show that the set $A = \{x\colon f^-(x) < f^+(x)\}$ is σ-porous.

The remaining cases are all handled in a similar fashion.

For every pair (r, s) of rational numbers satisfying $0 < r < s$, let

$$A_{rs} = \{x\colon f^-(x) < r < s < f^+(x)\}.$$

It then suffices to show that each A_{rs} is σ-porous. For each natural number n, let

$$A_{rsn} = \left\{x\colon x \in A_{rs} \text{ and } \frac{f(y) - f(x)}{y - x} < r \text{ for } x - 1/n < y < x\right\}.$$

We shall show that A_{rsn} is a porous set. Indeed, A_{rsn} will have porosity greater than or equal to $1 - r/s$ at each of its points.

To see this, let $x \in A_{rsn}$. Then there is a sequence of numbers x_k converging to x from the right such that

$$\frac{f(x_k) - f(x)}{x_k - x} > s.$$

Take k so large that $x_k - x < 1/n$, let $w_k = x + s(x_k - x)/r$, and let $I_k = [x_k, w_k]$. Then for $y \in I_k$ we have

$$\frac{x_k - x}{y - x} \geqslant \frac{r}{s},$$

and using this along with the monotonicity of f, it follows that

$$\frac{f(y) - f(x)}{y - x} \geqslant \frac{f(x_k) - f(x)}{x_k - x} \cdot \frac{x_k - x}{y - x} \geqslant r.$$

Consequently, $y \notin A_{rsn}$. Furthermore, for each k we have

$$\frac{|I_k|}{w_k - x} = 1 - \frac{r}{s},$$

and hence A_{rsn} has porosity at least $1 - r/s$ at x.

Using this result and the argument employed by Neugebauer [5], we obtain the following

COROLLARY 1. *For an arbitrary function f: $R \to R$ the set*

$$B = \{x\colon C_e^-(f, x) \neq C_e^+(f, x)\}$$

is σ-porous.

Next we present an application of Theorem 1 to other types of derivates. Let $f^s(x)$ denote the upper symmetric derivate of f at x, that is,

$$f^s(x) = \lim_{h \to 0^+} \sup \frac{f(x+h) - f(x-h)}{2h}.$$

In [1] it was shown that if $f\colon R \to R$ is continuous at a dense set of points, then $f^s(x) = \max\{f^+(x), f^-(x)\}$ except at a σ-porous set of points.

Further, let $f^+_{ap}(x), f^-_{ap}(x)$, and $f^s_{ap}(x)$ denote the upper approximate right, left, and symmetric derivates of f at x, respectively. L. Mišík [4] has shown that if f is monotone, then $f^+_{ap}(x) = f^+(x)$ and $f^-_{ap}(x) = f^-(x)$ pointwise. A very similar proof shows that $f^s_{ap}(x) = f^s(x)$ pointwise for a monotone function.

Combining Theorem 1 and the information in the previous two paragraphs, we obtain

COROLLARY 2. *If $f\colon R \to R$ is monotone, then except at those points x in a σ-porous set we have*

$$f^+(x) = f^-(x) = f^s(x) = f^s_{ap}(x) = f^-_{ap}(x) = f^+_{ap}(x).$$

3. The inequality of unilateral derivates. For purposes of clarity we will divide this section into two parts. In the first part we prove that given any perfect set K of measure zero, there is a continuous bounded variation function such that $K \subseteq E$ (Theorem 2). The remainder of this section is then devoted to generalizing Theorem 2 to first category sets K of measure zero.

As the actual construction used in Theorem 2 is somewhat complicated, we first describe a function defining process which will be used in an inductive manner in the final construction. Let $I = [u, v]$ be a closed interval and let K be a perfect set of measure zero contained in I such that both u and v are in K. In addition, let $\varepsilon > 0$ be specified. The function we wish to define will be zero except at enumerably many contiguous intervals of very small cumulative measure; the union of these intervals will be denoted by B. Another finite set of contiguous intervals of large cumulative measure is then specified; the union of these intervals is denoted by A. In addition, A and B are chosen so that the remainder of I consists of enumerably many closed intervals whose union is denoted by C. Specifically, let $A = \bigcup_{k=1}^{N} (a_k, a'_k)$ where each interval (a_k, a'_k) is contiguous to K, $a'_k < a_{k+1}$, $k = 1, 2, \ldots, N-1$, and $\Sigma(a'_k - a_k) > v - u - \varepsilon/4$. For notational convenience, let $u = a'_0$ and $v = a_{N+1}$. Now, in each interval $[a'_k, a_{k+1}]$, $k = 0, 1, \ldots, N$, there is a sequence of intervals $\{(b_{ki}, b'_{ki})\colon i = 1, 2, \ldots\}$ each contiguous to K such that $b'_{ki} < b_{k(i+1)}$, and $\lim_{i\to\infty} b_{ki} = a_{k+1}$. For notational convenience we let $b'_{k0} = a'_k$, and let

$$B = \bigcup_{k=0}^{N} \bigcup_{i=1}^{\infty} (b_{ki}, b'_{ki}) \quad \text{and} \quad C = I - A \cup B.$$

Define $F(I, K, \varepsilon, A, B) = F(I, K, \varepsilon)$ as follows: F is to be zero except at the intervals $[b_{ki}, b'_{ki}]$, $k = 0, 1, \ldots, N$, $i = 1, 2, \ldots$; on each interval $[b_{ki}, b'_{ki}]$ define F to be a quadratic function with zeros at both b_{ki} and b'_{ki}, and a maximum of

$(b_{ki} + b'_{ki} - 2b'_{k(i-1)})/2$. First note that F is continuous, $F(K) = \{0\}$, F is zero outside of I, and the variation of F is less than $\varepsilon/2$. Now, if $x \in K$ and x is neither the left endpoint of an interval in A nor the maximum of K, then there is a $y \in I$ such that $[f(y) - f(x)]/[y - x] > 1$. This "$y$" can be chosen to be the midpoint of the "B interval" immediately to the right of x.

Generalizing slightly, let I be the union of a finite or denumerable set of mutually exclusive closed intervals, $I = \cup I_n$. Suppose K is a set such that $K \cap I_n$ is a perfect set of measure zero and K contains both endpoints of I_n for every n. Suppose further that $\varepsilon > 0$ is specified. Define $F(I_n, K \cap I_n, \varepsilon/2^{n-1}, A_n, B_n) \equiv F(I_n, K \cap I_n, \varepsilon/2^{n-1})$ as above and let

$$F(I, K, \varepsilon)(x) = \begin{cases} F(I_n, K \cap I_n, \varepsilon/2^{n-1})(x) & \text{if } x \in I_n, \\ 0 \quad \text{if } x \notin I. \end{cases}$$

In this case the variation of F is less than $\Sigma \varepsilon/2^n < \varepsilon$. The bother of identifying the sets A, B, and C for such functions is necessitated by the induction argument used in the proof below.

THEOREM 2. *If $K \subset (a, b)$ is a perfect set of measure zero and $\varepsilon > 0$, then there is a continuous function of bounded variation, f: $R \to [0, +\infty)$ such that*

(1) $f(x) = 0$ *if either $x \in K$ or $x \notin (a, b)$,*

(2) $f^+(x) > 1$ *and* $f^-(x) < 0$ *for every $x \in K$,*

(3) *the total variation of f, $V(f)$, is less than ε, and*

(4) *f is differentiable at every point not in K.*

PROOF. Let $u = \min(K)$, $v = \max(K)$, and $I_1 = [u, v]$. Let $f_1 \equiv F(I_1, K, \varepsilon/4, A_1, B_1)$. Suppose now that f_n has been defined and is of the form $F(I_n, K, \varepsilon/2^{n+1}, A_n, B_n)$ for the union I_n of some family of mutually exclusive closed intervals. Let $I_{n+1} = C_n(C_n = I_n - A_n \cup B_n)$ and define $f_{n+1} \equiv F(I_{n+1}, K, \varepsilon/2^{n+2}, A_{n+1}, B_{n+1})$. This completes the induction and we define $f^* = \Sigma f_n$.

It is evident that f^* is continuous. And as $f^*(K) = 0$ and $f^*(x) > 0$ for each x, $(f^*)^-(x) < 0$ for every $x \in K$. Also, as the variation of f_n is less than $\varepsilon/2^{n+1}$, f^* is of bounded variation and the variation of f^* is less than $\varepsilon/2$. Let $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$ and suppose $x \in K$ where $x \neq v$ and x is not the left endpoint of an interval from either A or B. Such an x is the intersection of a nested set of intervals $C_n(x) \subseteq C_n$ and x is not the right endpoint of any of the $C_n(x)$ intervals. Consequently, there is a point $y_n \in C_n(x)$ such that $[f^*(y_n) - f^*(x)]/[y_n - x] > 1$. It follows that for such x, $(f^*)^+(x) > 1$. If x is a left endpoint of an interval in B, then since $F'_+(b_{ki}) > 1$ for each k and i, there is an n such that $(f_n)'_+(x) > 1$, whence $(f^*)^+(x) > 1$. We must alter f^* slightly in order that the upper right derivate exceeds 1 for v and for left endpoints of intervals in A, and the remainder of the proof is devoted to this task. First note that A is the union of those intervals contiguous to K which have been mapped to zero by f^*. Enumerate those intervals as $\{(a_n, b_n): n = 1, 2, \ldots\}$, and for notational convenience, let $v = a_0$ and $b = b_0$. For each interval, (a_n, b_n) $n = 0, 1, 2, \ldots,$ let $c_n \in (a_n, b_n)$ such that $c_n - a_n < \varepsilon/2^{n+2}$. Define a function g: $R \to [0, +\infty)$ to be zero except on the intervals

$[a_n, c_n]$ $n = 0, 1, 2, \ldots$; on each interval $[a_n, c_n]$ define g to be a cubic function with a simple zero at a_n, a double zero at c_n, and a maximum of $(c_n - a_n)/2$. This function g is complementary to f^* in the sense that $g^{-1}(0, +\infty) \cap f^{*-1}(0, +\infty) = \varnothing$. The desired function is then $f = f^* + g$. This completes the proof of Theorem 2.

THEOREM 3. *Let K be a set which is of measure zero, and of the first Baire category. Then there is a continuous function f of bounded variation such that $|f^+(x) - f^-(x)| > 1$ for every $x \in K$.*

PROOF. As K is of measure zero and the first Baire category, a standard set theoretic argument shows that there is a set of mutually exclusive perfect sets of measure zero, $\{K_n: n = 1, 2, \ldots\}$ such that $K \subseteq \bigcup_{n=1}^{\infty} K_n$. Let f_1 be a continuous nonnegative bounded variation function such that:

(1) $f_1(K_1) = \{0\}$,

(2) $f_1^+(x) > 1$ and $f_1^-(x) < 0$ for every $x \in K_1$,

(3) $V(f_1) < \frac{1}{2}$, and

(4) f_1 is differentiable at every point not in K_1.

Denote the set of intervals contiguous to K_1 by $\{(a_i, b_i): i = 1, 2, \ldots\}$. On each interval (a_i, b_i) define f_2 as follows:

(i) If $K_2 \cap (a_i, b_i) = \varnothing$, let $f_2(a_i, b_i) = \{0\}$.

(ii) If $K_2 \cap (a_i, b_i) \neq \varnothing$, then $K_2 \cap (a_i, b_i)$ is perfect and we define f_2 on (a_i, b_i) using Theorem 2 with $\varepsilon = 1/2^{i+2}$.

Finally, we define $f_2(K_1) = \{0\}$. Note that $V(f_2) < \frac{1}{4}$ so that the variation of $g = f_1 + f_2$ is less than $\frac{3}{4}$. Further, if $x \in K_1$, $g^+(x) > 1$ and $g^-(x) < 0$ and if $x \in K_2$, $g^+(x) > 1 + s$ and $g^-(x) < s$ where s is the derivative of f_1 at x.

We continue inductively to obtain for each n a function f_n of variation less than $1/2^n$. Let $f = \sum_{n=1}^{\infty} f_n$. If $x \in \bigcup_{n=1}^{\infty} K_n$, then there is a unique $N = N(x)$ such that $x \in K_N$. Now for $n > N$, $f_n(x) = 0$ and the construction entails that

$$(f_N + f_{N+1} + \ldots)^+(x) > 1, \quad (f_N + f_{N+1} + \ldots)^-(x) < 0,$$

and each of the functions $f_1, f_2, \ldots, f_{N-1}$ has a derivative at x. It follows that $|f^+(x) - f^-(x)| > 1$ for each such x, and as f is evidently continuous and of variation less than one, the proof is complete.

REFERENCES

1. C. L. Belna, M. J. Evans and P. D. Humke, *Symmetric and ordinary differentiation*, Proc. Amer. Math. Soc. **72** (1978), 261–267.

2. E. P. Dolženko, *Boundary properties of arbitrary functions*, Math. USSR-Izv. **1** (1967), 1–12.

3. M. Kulbacka, *Sur l'ensemble des points de l'asymetrie approximative*, Acta Sci. Math. (Szeged) **21** (1960), 90–95.

4. L. Mišík, *Über approximative derivierte Zahlen monotoner Funktionen*, Czechoslovak Math. J. **26** (101) (1976), 579–583.

5. C. J. Neugebauer, *A theorem on derivates*, Acta Sci. Math. (Szeged) **23** (1962), 79–81.

6. L. Zajíček, *On cluster sets of arbitrary functions*, Fund. Math. **83** (1974), 197–217.

7. ______, *Sets of σ-porosity and sets of σ-porosity (q)*, Časopis Pěst. Mat. **101** (1976), 350–359.

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