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A NORMAL FORM FOR A SPECIAL CLASS OF CURVATURE OPERATORS

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ABSTRACT. In the case of a 4-dimensional oriented inner product space Singer and Thorpe found a canonical form for a curvature operator which commutes with a generator of Λ^4, and used it to prove that the curvature function is completely determined by its critical point behavior. In dimension 5 we extend these results to curvature operators which commute with an element of Λ^4.

1. Introduction. The Riemannian sectional curvature of a Riemannian manifold M is a real valued function σ on the Grassmann bundle of tangent 2-planes of M. Although there exist many theorems relating the curvature of M to various topological and geometric properties of M, there is little known of a general nature about the behavior of σ itself.

In [3], Thorpe gave a simple characterization of the minimal and maximal sets of σ, when the dimension $M \leqslant 4$. However his characterization does not hold in higher dimensions, see [4].

Let V be a real n-dimensional vector space with inner product, $\langle\ ,\ \rangle$. It is well known that if T is a symmetric linear transformation on V, then the diagonalization or normal form of T is equivalent to the analysis of the critical point behavior of the function $v \to \langle Tv, v\rangle$ on the projective space of V. It is also known, [1], that the curvature tensor at a point of a Riemannian manifold is completely determined by the sectional curvature function σ on the Grassmann manifold of 2-planes at the point. It then seems natural to look for a "normal form" for the curvature tensor by analyzing the critical point behavior of the sectional curvature function σ.

In [2] Singer and Thorpe prove that the curvature tensor of a 4-dimensional oriented Einstein manifold is completely determined by the critical point behavior of σ. Working at a point p, of a 4-dimensional oriented Riemannian manifold M, they view the curvature as a symmetric linear operator on Λ^2 of the tangent space TM_p. When the operator commutes with a generator of $\Lambda^4(TM_p)$, they show it has a simple canonical form and use it to prove that the critical point behavior of σ determines the operator. In dimension 5 we extend these results to curvature operators which commute with an element of Λ^4; the curvature tensor of the 5 sphere or of the product of an Einstein space and a line are examples of such operators.

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2. Preliminaries. Let V be an n-dimensional real vector space with inner product $\langle\ ,\ \rangle$ and for $v \in V$ set $|v| = \sqrt{\langle v, v\rangle}$. For p an integer, $1 \leqslant p \leqslant n$, by $\Lambda^p(V)$ or Λ^p we mean the space of p-vectors of V. If $\{e_1, \ldots, e_n\}$ is a basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | i_1 < \cdots < i_p\}$ is a basis for Λ^p and it follows that Λ^p has dimension $\binom{n}{p}$. A p-vector w is called decomposable if $w = v_1 \wedge \cdots \wedge v_p$ where $v_1, \ldots, v_p \in V$. Hence, Λ^p has a basis of decomposable vectors. Thus, when defining an inner product on Λ^p it suffices to specify its values on decomposable p-vectors. We set $\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p\rangle = \det[\langle u_i, v_j\rangle]$ where $u_i, v_j \in V$. For $\xi \in \Lambda^2$ we set $\|\xi\| = \sqrt{\langle \xi, \xi\rangle}$. It follows that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | i_1 < \cdots < i_p\}$ is an orthonormal basis for Λ^p. Let G denote the Grassmann manifold of oriented 2-dimensional subspaces of V; we identify G with the submanifold of Λ^2 consisting of decomposable 2-vectors of length 1 by $P \to u \wedge v$ where $\{u, v\}$ is any oriented orthonormal basis for P.

Let V be an n-dimensional real inner product space. A curvature operator R is a symmetric linear transformation on $\Lambda^2(V)$. The space $\mathfrak{R}$ of all curvature operators has dimension $[\binom{n}{2}^2 + \binom{n}{2}]/2$ and inner product given by $\langle R, T\rangle = \operatorname{trace} R \circ T$ where $R, T \in \mathfrak{R}$. Given $R \in \mathfrak{R}$ its sectional curvature is the function $\sigma_R\colon G \to \mathbf{R}$ defined by $\sigma_R(P) = \langle RP, P\rangle$, $P \in G$. A 2-plane (2-vector) P is a critical plane of R if P is a critical point of σ_R.

3. The Bianchi identity and the Grassmann quadratic 2-relations. In this section we examine the space $\mathfrak{S}$ which is complementary in $\mathfrak{R}$ to the subspace $\mathfrak{B} = \{R \in \mathfrak{R} | R$ satisfies the Bianchi identity$\}$. We recall that $\mathfrak{S}$ is naturally isomorphic to Λ^4 and we exhibit the relationship between $\mathfrak{S}$ and the Grassmann quadratic 2-relations which are necessary and sufficient conditions for decomposability of elements in Λ^2. These results are well known and detailed proofs can be found in [3].

Given $R \in \mathfrak{R}$ we associate a 2-form on V with values in the vector space of skew symmetric endomorphisms of V by

$$\langle R(u, v)(w), x\rangle = \langle Ru \wedge v, w \wedge x\rangle, \qquad u, v, w, x \in V.$$

It is easily checked that this "association" is a vector space isomorphism.

Using this identification we define the Bianchi map $b\colon \mathfrak{R} \to \mathfrak{R}$; given $R \in \mathfrak{R}$ we set

$$[b(R)](u, v)(w) = R(u, v)(w) + R(v, w)(u) + R(w, u)(v).$$

It is easily checked that b is a linear map and so its kernel is a linear subspace of $\mathfrak{R}$ which we will denote by $\mathfrak{B}$.

Let $\mathfrak{S} = \mathfrak{B}^\perp$, the orthogonal complement of $\mathfrak{B}$ in $\mathfrak{R}$. For each $\varepsilon \in \Lambda^4$ we associate $S_\varepsilon \in \mathfrak{R}$ by $\langle S_\varepsilon \alpha, \beta\rangle = \langle \varepsilon, \alpha \wedge \beta\rangle$, where $\alpha, \beta \in \Lambda^2$.

Proposition 3.1. *The map $\varepsilon \to S_\varepsilon$ is an isomorphism of Λ^4 onto $\mathfrak{S}$. In fact $\varepsilon \to S_\varepsilon/\sqrt{6}$ is an isometry.*

Proposition 3.2. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. For $1 \leqslant i < j < k < l \leqslant n$ set $S_{ijkl} = S_{e_i \wedge e_j \wedge e_k \wedge e_l}$. If $\alpha \in \Lambda^2$, then α is decomposable if and only if $\langle S_{ijkl}\alpha, \alpha\rangle = 0$, $1 \leqslant i < j < k < l \leqslant n$.*

REMARK 1. *The conditions* $\langle S_{ijkl}\alpha, \alpha\rangle = 0$, $1 \leq i < j < k < l \leq n$, *are known as the Grassmann quadratic* 2-*relations.*

REMARK 2. *In view of Proposition* 3.2 *it is clear that each curvature operator* $S \in \mathbb{S}$ *has sectional curvature* σ_S *identically zero. Conversely, it is easily checked that this property characterizes* $\mathbb{S}$.

4. Critical planes and normal forms. In this section we restrict ourselves to the case where dimension $V = 5$. Letting $\{e_1, \ldots, e_5\}$ be an orthonormal basis for V we use the operators S_{ijkl}, $1 \leq i < j < k < l \leq 5$, defined in §3, to derive necessary and sufficient conditions for $P \in G$ to be a critical plane of a curvature operator R. Next we derive a normal form for a special class of curvature operators and use this normal form to prove that these operators are determined uniquely by their critical planes and values of sectional curvature on these critical planes.

Set $J = \{(i, j, k, l): 1 \leq i < j < k < l \leq 5\}$.

PROPOSITION 4.1. *$P \in G$ is a critical plane of R if and only if*

$$RP = \lambda P + \sum_J \mu_{ijkl} S_{ijkl} P$$

for some real numbers λ and μ_{ijkl}, $(i, j, k, l) \in J$. The number λ is the (critical) value of σ_R at P.

PROOF. Consider the real valued functions defined on Λ^2 by $f(\xi) = \langle R\xi, \xi\rangle$, $g(\xi) = \langle \xi, \xi\rangle$, and $h_{ijkl}(\xi) = \langle S_{ijkl}\xi, \xi\rangle$, $1 \leq i < j < k < l \leq 5$. Then σ is the restriction of f to $G = g^{-1}(1) \cap_J h_{ijkl}^{-1}(0)$. $P \in G$ is a critical point of σ_R if and only if at P we have

$$\operatorname{grad} f = \lambda \operatorname{grad} g + \sum_J \mu_{ijkl} S_{ijkl} P$$

where λ and μ_{ijkl}, $(i, j, k, l) \in J$, are Lagrange multipliers. Now for a quadratic form $\langle A\xi, \xi\rangle$, the gradient at P is just $2AP$. Applying this to the quadratic forms f, g and h_{ijkl}, $1 \leq i < j < k < l \leq 5$, completes the proof.

THEOREM 4.2. *Let $R \in \mathfrak{R}$ and suppose that for some $\eta \in \Lambda^4$, $\eta \neq 0$, $RS_\eta = S_\eta R$. Then there exists an orthonormal basis for Λ^2 consisting of critical planes of R and relative to this basis the matrix for R has the form*

$$[R] = \left[\begin{array}{cc|c} A & B & 0 \\ B & A & \\ \hline \multicolumn{2}{c|}{0} & C \end{array}\right]$$

where

$$A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{bmatrix}$$

and

$$C = \begin{bmatrix} a_7 & & & \\ & a_8 & & \\ & & a_9 & \\ & & & a_{10} \end{bmatrix}.$$

PROOF. We note that since dimension $V = 5$ each element of $\Lambda^4(V)$ is decomposable. Let $\{e_1, \ldots, e_5\}$ be an orthonormal basis for V such that $\eta/\|\eta\| = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Since $RS_\eta = S_\eta R$ the symmetric operators R and S_η can be simultaneously diagonalized. Hence we can let $\{\xi_1, \ldots, \xi_{10}\}$ be an orthonormal basis for Λ^2 such that $S_\eta \xi_i = \xi_i$ $(1 \leqslant i \leqslant 3)$, $S_\eta \xi_i = -\xi_i$ $(4 \leqslant i \leqslant 6)$, $S_\eta \xi_i = 0$ $(7 \leqslant i \leqslant 10)$, and $R\xi_i = a_i \xi_i$ $(1 \leqslant i \leqslant 10)$. It is easily checked that $\{e_1 \wedge e_5, e_2 \wedge e_5, e_3 \wedge e_5, e_4 \wedge e_5\}$ is a basis for the zero eigenspace of S_η and then that each eigenvector with eigenvalue zero (in particular ξ_7, ξ_8, ξ_9, and ξ_{10}) is decomposable.

Let $W = \operatorname{Span}\{e_1, e_2, e_3, e_4\}$ and consider

$$P_i = \begin{cases} (\xi_i + \xi_{i+3})/\sqrt{2}\,, & 1 \leqslant i \leqslant 3, \\ (\xi_{i-3} - \xi_i)/\sqrt{2}\,, & 4 \leqslant i \leqslant 6. \end{cases}$$

It is readily checked that $\langle S_\eta P_i, P_i \rangle = 0$, and since $P_i \in \Lambda^2(W)$, $1 \leqslant i \leqslant 6$, it follows by Proposition 3.2 that each P_i is decomposable. Now $R\xi_i = a_i \xi_i$ $(7 \leqslant i \leqslant 10)$ and a short computation shows that $RP_i = \lambda_i P_i + \mu_i S_\eta P_i$, $1 \leqslant i \leqslant 6$, where $\lambda_i = \lambda_{i+3} = (a_i + a_{i+3})/2$, $1 \leqslant i \leqslant 3$, and $\mu_i = \mu_{i+3} = (a_i - a_{i+3})/2$, $1 \leqslant i \leqslant 3$. Hence it follows by Proposition 4.1 that $\{P_1, P_2, P_3, P_4, P_5, P_6, \xi_7, \xi_8, \xi_9 \xi_{10}\}$ is an orthonormal basis for Λ^2 consisting of critical planes of R. The matrix $[R]$ with respect to this basis has the required form.

REMARK 4.3. The above normal form is not unique. In fact an examination of the proof of Theorem 4.2 shows that (neglecting orientations) when the eigenvalues of R are distinct there are nine distinct critical plane pairs (P_{ij}^+, P_{ij}^-) one corresponding to each pair (ξ_i, ξ_{j+3}) $(i, j \leqslant 3)$ of eigenvectors of R. Since $P_{ij}^\pm$ and $P_{kl}^\pm$ are orthogonal if and only if $i \neq k$ and $j \neq l$, it follows that there are six distinct normal forms. (We make no distinction between two matrices which differ by permutations of rows and columns.)

When the eigenvalues of R are not distinct, then the critical values of σ may not be distinct and we may get less than six distinct normal forms.

THEOREM 4.4. *Suppose $R \in B$ and that for some $\eta \in \Lambda^4$, $\eta \neq 0$, $RS_\eta = S_\eta R$. Then R is completely determined by its critical planes and the values of the sectional curvature on these planes.*

PROOF. Let $P_{ij}^\pm$ $(i, j \leqslant 3)$ and ξ_i $(6 \leqslant i \leqslant 10)$ be the critical planes in the proof of Theorem 4.2 and Remark 4.3 Let $\lambda_{ij} = \sigma(P_{ij}^\pm)$. Then $RP_{ij}^\pm = \lambda_{ij} P_{ij}^\pm + \mu_{ij} S_\eta P_{ij}^\pm$ where $(\lambda_{ij}, \mu_{ij}) = ([a_i + a_{j+3}]/2, [a_i - a_{j+3}]/2$. Since $R \in \mathfrak{B}$, $0 = \langle R, S_\eta \rangle = RS_\eta = a_1 + a_2 + a_3 - a_4 - a_5 - a_6$, and a short computation shows that $\mu_{ij} = (\lambda_{ij'} + \lambda_{ij''} - \lambda_{i'j} - \lambda_{i''j})/3$ where (i, i', i'') and (j, j', j'') are arbitrary permutations of $(1, 2, 3)$. Thus, the matrix for R (with respect to the basis $\{P_{11}^+, P_{22}^+, P_{33}^+, P_{11}^-, P_{22}^-, P_{33}^-, \xi_7, \xi_8, \xi_9, \xi_{10}\}$) and so R itself is completely determined by the λ_{ij}'s.

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