SLICING CONVEX BODIES—BOUNDS FOR SLICE AREA 
IN TERMS OF THE BODY'S COVARIANCE

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ABSTRACT. Let $\mathcal{Q}$ be a zero-symmetric convex set in $\mathbb{R}^N$ with volume 1 and 
covariance matrix $V^2\text{Id}_{N \times N}$. Let $P$ be a $K$-dimensional vector subspace of $\mathbb{R}^N$, $K < N$, 
and let $J = N - K$. Then there exist constants $C_1(J)$ and $C_2(J)$ such that 
$$\nu^{-J}C_1(J) < \text{vol}_K(P \cap \mathcal{Q}) < \nu^{-J}C_2(J).$$

The lower bound has applications to Diophantine equations.

1. Introduction. The restriction to bodies of covariance a constant multiple of 
$\text{Id}_{N \times N}$ and volume 1 made in the abstract is not essential, as any centro-symmetric 
convex body can be brought to that form by a suitable linear transformation. Yet 
such bodies comprise the most important special cases. The unit cube, the $L^1$ ball 
$\sum_i |x_i| \leq r$ of volume 1 and the “complex cube” $|z_i| \leq \pi^{-1/2}$, $1 \leq i < N$, are 
examples.

For the case of the cube, real or complex, only the upper bound is of interest as 
there is a sharp lower bound of 1, independent of $K$ and $N$, due to Vaaler \cite{10}. For 
the real cube in the case $K = N - 1$ an upper bound of 5 was given in \cite{6}, which 
we improve here to $\sqrt{6}$.

Examples with a cube show that $C_1(J) \geq 12^{-J/2}$ and $C_2(J) \leq 6^{-J/2}$. We take 
$$C_1(J) = (J + 2)^{-J/2} - 3/2 \Gamma(1/2J + 1),$$

$$C_2(J) = 2(8(\log 2)^{-J/2}(J + 2))^{J/2} \Gamma(1/2J + 1)$$

for $J \geq 2$, and $C_2(1) = 1/\sqrt{2}$. Then we have

**Theorem 1.** Let $\mathcal{Q}$ be a centro-symmetric convex body in $\mathbb{R}^N$ with volume $U$ and 
diagonal covariance matrix $(V_{ij}^2 \delta_{ij})$, $1 \leq i, j \leq N$. Let $P$ be a $K$-dimensional vector 
subspace of $\mathbb{R}^N$ with $K < N$, $K + J = N$. Let 
$$V = \left( U^{-N - 2} \prod_{i=1}^{N} V_i^2 \right)^{1/2N}$$

and $c_i = V_i V^{-1} U^{-1/2}$.

Let $W_K$ be the product of the $K$ smallest $c_i$ and $W'_K$ the product of the $K$ largest $c_i$. 

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The lower bound 1 for the case $K = N - 1$ is implicit in H. Hadwiger’s *Gitterperiodische Punktmengen 

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Then

$$C_1(J)W_KV^{-J} < \text{vol}_K(P \cap Q) < C_2(J)W_K'V^{-J}.$$ 

If all the $V_i$ are equal to some $V$, and $U = 1$ then all $c_i$, $W_K$, and $W_K'$ are 1 and Theorem 1 reduces to

**Theorem 1'.** Let $Q$ be a centro-symmetric convex body in $\mathbb{R}^N$ with volume 1 and covariance matrix $V^2\text{Id}_{N \times N}$. Let $P$ be a $K$-dimensional vector subspace of $\mathbb{R}^N$, $K < N$ and $J = N - K$. Then

$$V^{-J}C_1(J) < \text{vol}_K(P \cap Q) < V^{-J}C_2(J).$$

Theorem 1 follows from 1' by an elementary lemma whose proof we omit. (But see Example 12, [8].)

**Lemma 1.** Let $E$ be a convex set in $\mathbb{R}^N$ of dimension $K < N$. Let $T$ be a linear transformation which maps each unit coordinate vector $\hat{e}_i$, $1 < i < N$, in $\mathbb{R}^N$ to $c_ie_i$, with $c_i > 0$. Let $E' = TE$. Let $W_K$ be the product of the $K$ smallest $c_i$ and $W_K'$ the product of the $K$ largest $c_i$. Then

$$W_K\text{vol}_KE < \text{vol}_KE' < W_K'\text{vol}_KE.$$

One may adapt our lower bound to Vaaler [12] by using a variant of his Lemma 6. Suppose $Q \subseteq \mathbb{R}^N$ has volume 1, is centro-symmetric and has covariance matrix $\text{Cov}(Q) = V^2\text{Id}_{N \times N}$. Let $L_j(x)$, $1 < j < N$, be $N$ real linear forms in $K$ variables, $L_j(x) = \sum_{k=1}^{K} a_{jk}x_k$, so that $A = (a_{jk})$ is an $N \times K$ matrix, with $N > K$, $N = K + J$.

**Lemma 6'.** Let $M$ be a positive integer and suppose that

$$M\left|\det A^TA\right|^{1/2} < V^{-J}C_1(J).$$

Then there exist at least $M$ distinct pairs of nonzero lattice points $\pm \bar{e}_m$, $1 < m < M$, such that for each $m$,

$$\bar{I}_m = (L_j(\bar{e}_m)) \in 2Q.$$ 

With Lemma 6' in place of Lemma 6 of [12] and following [12] otherwise, we have a generalization of that paper's main result. Let $\Lambda_j(\bar{x}) = \sum_{k=1}^{K} a_{jk}x_k$ for $1 < j < J$ be $J$ real linear forms in $K$ real variables $x_1, \ldots, x_K$. Let $N = J + K$. Let $Q$ be centro-symmetric and convex with $\text{vol}_NQ = 1$ and $\text{Cov}(Q) = V^2\text{Id}_{N \times N}$. For $\bar{y} \in \mathbb{R}^K$ let $l(\bar{y}) = (y_1, y_2, \ldots, y_K, \Lambda_1(\bar{y}), \ldots, \Lambda_J(\bar{y})) \in \mathbb{R}^N$.

**Theorem 2.** Let $M$ be a positive integer and suppose that

$$M^2 \prod_{1}^{J}\left(1 + \sum_{1}^{K}|a_{jk}|^2\right) < C_1(J)V^{-J}.$$ 

Then there exist $M$ distinct pairs of nonzero lattice points $\pm \bar{e}_m$, $1 < m < M$, in $\mathbb{Z}^K$ such that $l(\pm \bar{e}_m) \in 2Q$. 

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Remark. When $Q$ is taken to be the unit cube we get a weaker version of Theorem 1 in [12] with all $a_k$ and $\beta_p$ there being 1; our more general lower bound is not as sharp for the cube as that in [11].


For the proof of Theorem 1 from Theorem 1' we take the $c_i$ in the statement of Theorem 1 as defining $T$, and let $Q' = T^{-1}Q$, $P' = T^{-1}P$. Applying Theorem 1' to $Q'$ and then Lemma 1 to the resulting bounds on $\text{vol}_K(P \cap Q')$ gives Theorem 1.

It remains to prove Theorem 1'. For the proof we shall need some lemmas about centro-symmetric log concave functions. (A function $f: \mathbb{R}^r \to \mathbb{R}^+ \cup \{0\}$ is log concave if $\log f$ is concave.)

**Lemma 2.** Suppose $f: \mathbb{R}^r \to \mathbb{R}^+ \cup \{0\}$ satisfies $f(x) = f(-\hat{\theta}c)$, $\{\hat{\theta}: f(x) > t\}$ is convex and open for each $t$, and $\int_{\mathbb{R}^r} f(x) d^r(\hat{\theta}) = 1$. Suppose further that for all unit $\hat{\theta} \in \mathbb{R}^r$,

$$\int_{\mathbb{R}^r} (\hat{\theta} \cdot \hat{\theta})^2 f(\hat{\theta}) \, d^r(\hat{\theta}) < V^2.$$

Then

$$f(0) > V^{-J}(J + 2)^{-J/2} \pi^{J/2} J (J + 1) = V^{-J} C_J(J).$$

**Proof.** Let $h: \mathbb{R}^r \to \mathbb{R}^+ \cup \{0\}$ be constant at $V^{-J} C_J(J)$ for $||\hat{\theta}|| < R = \sqrt{V(J + 2)^{1/2}}$, and 0 for $||\hat{\theta}|| > R$. Then $\int_{\mathbb{R}^r} h = 1$ and $\int_{\mathbb{R}^r} x^2 h = V^2$, so $h$ satisfies the hypotheses of Lemma 2. Thus to prove the lemma it suffices to prove that if $f \neq h$, $f(0) > h(0)$.

We first show that $f$ can be replaced by an $f_1$ such that $f_1$ is circular symmetric, that is, $f_1(\hat{x}) = f_1(\hat{y})$ if $||\hat{x}|| = ||\hat{y}||$, and such that $f_1(\hat{0}) = f(\hat{0})$ and $f_1$ satisfies the hypotheses of the lemma.

For let $E_t = \{\hat{x}: f(\hat{x}) > t\}$. By hypothesis $E_t$ is convex, and if $t < s$, $E_t \subseteq E_s$. Thus

$$\int_{\mathbb{R}^r} f(\hat{x}) = \int_{t=0}^{f(0)} \int_{E_t} 1 \, d^r(\hat{x}) \, dt, \quad (3)$$

while

$$\int_{\mathbb{R}^r} x^2 f(\hat{x}) = \int_{t=0}^{f(0)} \int_{E_t} x^2 \, d^r(\hat{x}) \, dt. \quad (4)$$

Let $E'_t$ be the ball about $\hat{0}$ of the same vol, as $E_t$, and let $f_1(\hat{x}) = \sup\{t: \hat{x} \in E'_t\}$. Then

$$\int_{\mathbb{R}^r} f_1(\hat{x}) = \int_{0}^{f(0)} \int_{E'_t} 1 \, d^r(\hat{x}) \, dt = \int_{E_t} f(\hat{x}) = 1. \quad (5)$$

Further,

$$\int_{\mathbb{R}^r} x^2 f_1(\hat{x}) = J^{-1} \int_{\mathbb{R}^r} ||\hat{x}||^2 f_1(\hat{x}) = J^{-1} \int_{E'_t} ||\hat{x}||^2 \, d^r(\hat{x}) \, dt. \quad (6)$$
Now consider a particular value of \( t \). If we show that \( \| x \|^2 < \| \overline{x} \|^2 \), the claim about \( f \) is proved. So let \( \text{INT} = E_t' \cap E_t \), \( \text{EX} = E_t \setminus E_t' \) and \( \text{EX}' = E_t' \setminus E_t \). Then

\[
\int_{E_t} \| x \|^2 = \int_{\text{INT}} \| x \|^2 + \int_{\text{EX}} \| x \|^2 > \int_{\text{INT}} \| \overline{x} \|^2 + \int_{\text{EX}'} \| \overline{x} \|^2 = \int_{E_t} \| \overline{x} \|^2,
\]

because \( \| \overline{x} \|^2 \) is everywhere in \( \text{EX} \) at least as large as anywhere in \( \text{EX}' \) and \( \text{vol}_{\overline{J}} \text{EX} = \text{vol}_{\overline{J}} \text{EX}' \). Thus \( \int_{\overline{R}^d} \| x \|^2 f_t(x) \, d\overline{x} < \int_{\overline{R}^d} \| \overline{x} \|^2 f_{\overline{t}}(\overline{x}) \, d\overline{x} \), and because of the circular symmetry of \( f_1 \), \( \int_{\overline{R}^d} (\overline{x} \cdot \theta) \overline{f}_t(\overline{x}) \, d\overline{x} \) \( < V^2 \) for all unit vectors \( \theta \).

Now suppose the lemma is false and \( h(\overline{0}) > f(\overline{0}) = f_1(\overline{0}) \). Since \( h \) and \( f_1 \) depend only on \( \| x \| = r \) we shall by an abuse of language write \( h(r), f_1(r) \) for \( h(\overline{x}), f_1(\overline{x}) \) when \( \| x \| = r \). (For \( r < R \), \( h(r) = h(0), \) as well.) Now

\[
\int_{\overline{R}^d} f_t(x) \, d\overline{x} = \int_0^\infty f_t(r) \sigma J_1 r^{-1} \, dr = 1 = \int_{\overline{R}^d} h(x) \, d\overline{x}.
\]

Let \( F_1(r) = \int_0^r f_1(u) \sigma J_1 u^{-1} \, du \) and \( H(r) = \int_0^r h(u) \sigma J_1 u^{-1} \, du \). Since for \( r < R \), \( h(0) = h(r) > f_1(r) \) while for \( r > R \), \( 0 = h(r) < f_1(r) \), we have \( H(r) > F_1(r) \) for \( 0 < r < R \), and \( H(r) > F_1(r) \) in any case. Thus

\[
\int_0^\infty 2r F_1(r) \, dr < \int_0^\infty 2r H(r) \, dr.
\]

and

\[
\int_0^\infty r^2 F_1(r) \, dr = \int_0^\infty r^2 f_1(r) \sigma J_1 r^{-1} \, dr > \int_0^\infty r^2 H(r) = \int_0^R r^2 h(0) \sigma J_1 r^{-1} \, dr.
\]

In other words,

\[
\int_{\overline{R}^d} \| \overline{x} \|^2 f_t(\overline{x}) \, d\overline{x} > \int_{\overline{R}^d} \| \overline{x} \|^2 h(\overline{x}) \, d\overline{x},
\]

a contradiction. □

Remark. Log concave centro-symmetric functions which satisfy the covariance hypothesis satisfy the other hypotheses of Lemma 2, as is proved in the preliminary lemma of [2]. Lemma 2 gives a sharp lower bound since an extremal function, \( h \), is found. Our next lemma is not as sharp.

**Lemma 3.** Suppose \( f: \overline{R}^d \to \overline{R}^+ \cup \{0\} \) satisfies \( f(\overline{x}) = f(-\overline{x}) \), is log concave, and \( \int_{\overline{R}^d} f(\overline{x}) \, d\overline{x} = 1 \). Suppose further that for all unit \( \theta \in \overline{R}^d \),

\[
\int_{\overline{R}^d} (\overline{x} \cdot \theta) f_{\overline{t}}(\overline{x}) \, d\overline{x} > V^2 .
\]

Then \( f(\overline{0}) < V^{-2} C_2(\overline{J}) \), where \( C_2(\overline{J}) \) is the same as in Theorem 1.

**Proof.** Let \( E_1 = \{ \overline{x} \in \overline{R}^d: f(\overline{x}) > \frac{1}{2} f(\overline{0}) \} \). \( E_1 \) is convex and centro-symmetric. Let \( E_i = iE_1 \setminus (i-1)E_1 \) for \( i \), so that \( \overline{R}^d \) is the disjoint union

\[
\bigcup_{i=1}^\infty E_i.
\]

Let \( r \) be the minimal radius of \( E_1 \), and let \( \theta \) be a unit vector in the direction of a point on \( \partial E_1 \) of norm \( r \).
Since $\int_{\mathbb{R}^n} (x \cdot \bar{\theta})^2 f(x) \, dx > V^2$,

$$V^2 \leq \sum_{i=1}^{\infty} \int_{E_i} (x \cdot \bar{\theta})^2 f(x) \, dx < 4 \sum_{i=1}^{\infty} i^{j+2s-2} \rho^2$$

because $\text{vol}_j E_i < 2 / f(0)$ and $f(0) \int_{E_i} (x \cdot \bar{\theta})^2 f(x) \, dx < (2 / f(0)) r^2 f(0) = 2 r^2$. Now

$$4 \sum_{i=1}^{\infty} i^{j+2s-2} \rho^2 < 8 r^2 \int_0^{\infty} s^{j+2s-2} \, ds = Dr^2 = 8 r^2 (\log 2)^{-j-3} (J + 2)!$$

by the integral comparison test. Thus $r > D^{-1/2} V$. On the other hand $2 > f(0) \text{vol}_j (E_i) > f(0) \sigma_j r^j$ so

$$f(0) < 2 r^{-j} \sigma_j^{-1} < 2 V^{-j} r^{j-1}$$

$$= V^{-j} \cdot 2 (8 (\log 2)^{-j-3} (J + 2)!)^{1/2} r^{j/2} \Gamma \left( \frac{1}{2} J + 1 \right) = V^{-j} C_2(J). \quad \square$$

In case $J = 1$ we can find the extremal function and obtain a sharper upper bound.

**Lemma 4.** Suppose $f : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$ satisfies $f(x) = f(-x)$, is log concave, and $\int_{\mathbb{R}} f(x) \, dx = 1$, $\int_{\mathbb{R}} x^2 f(x) \, dx = V^2$. Then $f(0) > 2^{-1/2} V^{-1}$.

**Proof.** Let $a = f(0)$. Since $\int_{-\infty}^{\infty} e^{-2a|t|} \, dt = 1$, $f(x)$ cannot be everywhere $> ae^{-2a|x|}$. They are equal at 0, so either $f(x) \equiv ae^{-2a|x|}$ or there exists $\beta > 0$ such that for $0 < x < \beta, f(x) > ae^{-2a|x|}$ while for $x > \beta, f(x) < ae^{-2a|x|}$. (Both are log concave.)

Now let $F(x) = \int_0^x f(t) \, dt$ and $G(x) = \int_0^x ae^{-2a|t|} \, dt$. Then

$$\int_{-\infty}^{\infty} x^2 (f(x) - ae^{-2a|x|}) \, dx = 2 \int_0^{\infty} x^2 (f(x) - ae^{-2ax}) \, dx$$

$$= 4 \int_0^{\infty} x (G(x) - F(x)) \, dx$$

since $x^2 (F - G)$ declines exponentially to zero at $\pm \infty$. Since $F(0) = G(0) = 0$ and $F(\infty) = G(\infty) = \frac{1}{2}$, and since $d(F - G) / dx > 0$ only for $0 < x < \beta$, $x(G(x) - F(x))$ is always negative and $\int_{-\infty}^{\infty} x^2 (f(x) - ae^{-2a|x|}) \, dx < 0$. \quad \square

What does all this have to do with $\text{vol}_N (P \cap Q)$? Let $K_J$ denote the unit cube $|x_i| < \frac{1}{2}$, $K < i < N$. Let $B$ denote the null matrix of $P$, so that $B$ has $N$ columns and $J$ rows, and $B \bar{x} = \vec{0}$ if $\bar{x} \in P$. Without loss of generality we may assume rank$(B) = J$. Let $B_\varepsilon = [B | \varepsilon d_J]$, and let $P_\varepsilon$ be the null space of $B_\varepsilon$. Then

$$\lim_{\varepsilon \to 0} \text{vol}_N (P_\varepsilon \cap (Q \times K_J)) = \text{vol}_N (P \cap Q), \quad (7)$$

and

$$\text{vol}_N (P_\varepsilon \cap (Q \times K_J)) = \text{vol}_N \{ \bar{z} : B_\varepsilon \bar{z} = 0 \text{ and } \bar{z} \in Q \times K_J \}. \quad (8)$$

We wish to compare the volume in $(8)$ to its projection onto the first $N$ coordinates, $\text{vol}_N (\bar{x} \in Q : B \bar{x} \in \varepsilon K_J)$. This ratio is $\varepsilon^{-J} (\det B_\varepsilon B_\varepsilon^T)^{1/2}$, which we now prove.

We shall need a lemma about how areas are affected by projection. If $\bar{a}_1 \ldots \bar{a}_N$ are column vectors in $\mathbb{R}^{N+J}$, if $A = [\bar{a}_1 \bar{a}_2 \ldots \bar{a}_N]^T$ is a matrix of $N + J$ columns
and $N$ rows, and $\text{Box}_A = \{ \tilde{a} : \tilde{a} = \sum_{i}^{N} \lambda_i \tilde{a}_i, 0 < \lambda_i < 1 \}$ then $\text{vol}_\nu \text{Box}_A = (\det AA^T)^{1/2}$ (Eves [3]).

Now suppose $A$ has the form $[\text{Id}_{N \times N} - D]$ and let $A' = [D^T \text{Id}_{J \times J}]$.

**Lemma 5.** $\det(AA^T) = \det(A'A'^T)$.

**Proof.** By the Cauchy-Binet theorem, $\det(AA^T) = \sum_\alpha (\det \alpha)^2$ where the summation is over all $N \times N$ minors $\alpha$ of $A$ [5]. Now a typical $\alpha$ consists of $l$ columns of $\text{Id}_{N \times N}$ and $m$ columns of $D$, $l < N$, $m < J$, $l + m = N$.

$$\alpha = \left( \tilde{e}_{a_1}, \tilde{e}_{a_2}, \ldots, \tilde{e}_{a_l} - \tilde{D}_{a_{l+1}}, \ldots, \tilde{e}_{a_{l+m}} - \tilde{D}_{a_{l+m}} \right).$$

Expanding $\det \alpha$ about the 1st through $l$th columns, each of which contain 1 once and otherwise zeros, we have $\det(\alpha) = \pm \det(\beta)$ where $\beta$ is the square matrix consisting of the intersection of the $N - l$ rows not indexed by $a_1, a_2, \ldots, a_l$ and the $m$ columns of $-D$ belonging to $\alpha$. Thus $\det(AA^T) = 1 + \sum_\beta (\det \beta)^2$, the sum taken over all $m \times m$ minors of $D$, $m < J$. But $\det(A'A'^T) = \sum_\gamma (\det \gamma)^2$, sum over $J \times J$ minors $\gamma$ of $A'$, and these may be expanded about the columns of $\text{Id}_{J \times J}$ to obtain again $1 + \sum_\beta (\det \beta)^2$. \qed

Now let $\tilde{b}_i, 1 < i < N$, denote the column vectors of $B$. Let

$$\tilde{a}_i = \left[ \begin{array}{c} \tilde{e}_i \\ -e^{-1}\tilde{b}_i \end{array} \right]$$

and

$$A^T = \left[ \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_N \right].$$

Then $B^T A^T = (0_{J \times N})$ so that the $\tilde{a}_i$ are vectors in $\text{Null}(B)$. When these $\tilde{a}_i$ are projected onto their first $N$ coordinates they are $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_N$. Therefore they are linearly independent and form a basis of $\text{Null}(B) = P_e$.

Now $\text{vol}_\nu \text{Box}_A = (\det AA^T)^{1/2}$ while $\text{vol}_\nu \text{Proj}(\text{Box}_A) = 1$ as $\tilde{e}_1, \ldots, \tilde{e}_N$ are orthonormal. The ratio of volumes, $(\det AA^T)^{1/2}$, is independent of which measurable set is projected.

Now $A$ has the form $A = [\text{Id}_{N \times N} - D]$ with $D = e^{-1}B^T$, so that $A' = [e^{-1}B] \text{Id}_{J \times J}$. Thus

$$\left( \det AA^T \right)^{1/2} = \left( \det A'A'^T \right)^{1/2}$$

by Lemma 5.

$$= e^{-J}(\det B_e B_e^T)^{1/2},$$

and this is the ratio of a volume in $P_e$ to the volume of its projection onto the first $N$ coordinates, as claimed.

Since we can find $\text{vol}_\nu(P_e \cap (Q \times K_j))$ from $\text{vol}_\nu(\{x \in Q : Bx \in eK_j\})$, we turn our attention to this last. It may be regarded as the probability that an $x$ taken "at random" from $Q$ (the probability measure being Lebesgue measure restricted to $Q$) will satisfy $Bx \in eK_j$. Let $f(x)$ denote the probability density function of $Bx$. Since $Q$ is convex, $f(x)$ is log concave. (This is the key observation.)

For let $\mu$ denote the probability measure associated with $f$, and $\nu$ Lebesgue measure restricted to $Q$. Let $s = 1 - t, 0 < t < 1$, and let $C, D$ be open convex sets in $\mathbb{R}^t$. Let $B^{-1}C, B^{-1}D$ be the inverse images in $\mathbb{R}^N$ under $B$ of $C$ and $D$ respectively. From Prekópa [9], since $\chi_Q$, the characteristic function of $Q$, is log concave, the measure $\nu$ is log concave, that is, $\nu(sC' + tD') > (\nu(C'))^s(\nu(D'))^t$. Let
C' = B'^{-1}D, D' = B'^{-1}D. Then
\[ \mu(sC + tD) = \nu(B^{-1}(sC + tD)) = \nu(sB^{-1}C + tB^{-1}D) \] (since B is linear)
\[ = \nu(sC' + tD') > (\nu(C'))^t(\nu(D'))^t = (\mu(C))^t(\mu(D))^t. \]
So \( \mu \) is a log concave measure and, again by Prekópa [9], \( f \) is a log concave function. Now as \( \varepsilon \to 0 \),
\[ \text{Prob}(B\bar{x} \in \varepsilon K_j) \sim e^J f(\bar{0}). \] (9)
Also,
\[ \text{Cov}(f)_{ij} = \int_{\mathbb{R}^J} x_i x_j f(\bar{x}) d\mu(\bar{x}) = (V^2BB^T)_{ij}. \] (10)
Let \( \bar{X} \) be the random vector uniformly distributed on \( Q \). Since \( BB^T \) is selfadjoint and positive definite (rank \( B = J \)), there is a square matrix \( S \) such that \( SS^T = BB^T \) [1].
Let \( Y \) be the random vector \( \bar{Y} = S^{-1}B\bar{X} \). Then \( \text{Cov}(B\bar{X}) = V^2BB^T \), so \( \text{Cov}(\bar{Y}) = V^2S^{-1}BB^TS^{-1T} = V^2\text{Id}_J \), since for any random vector \( \bar{Z} \), any square matrix \( U \), \( \text{Cov}(U\bar{Z}) = E(U\bar{Z}(U\bar{Z})^T) = E(U\bar{Z}\bar{Z}^TU^T) = U \text{Cov}(\bar{Z})U^T = U \text{Cov}(\bar{Y})U^T \) because expectation (7\( \bar{E} \)) is linear.
Let \( h(\bar{x}) \) be the probability density function associated with \( \bar{Y} \). Then
\[ h(\bar{0}) = (\det BB^T)^{1/2} f(\bar{0}), \] (11)
\[ \text{Cov}(h) = \text{Cov}(\bar{Y}) = V^2\text{Id}_J, \] (12)
and since log concavity is preserved under the linear transformation \( S^{-1} \), \( h \) is log concave.
Applying Lemmas 2 and 3 to \( h \), and Lemma 4 in case \( J = 1 \), we have from (12) that
\[ C_1(J)V^{-J} < h(\bar{0}) < C_2(J)V^{-J} \] (13)
and from (7) through (11) that
\[ C_1(J)V^{-J} < \text{vol}_K(P \cap Q) < C_2(J)V^{-J}. \]

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