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THE EXISTENCE OF Q-SETS IS EQUIVALENT TO THE EXISTENCE OF STRONG Q-SETS

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ABSTRACT. In this note we prove that the existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set, i.e. a Q-set all finite powers of which are Q-sets.

A *Q-set* is a separable metric space all subsets of which are F_σ sets. It is well known that the existence of an uncountable Q-set is equivalent to the existence of a normal separable nonmetrizable Moore space and is undecidable in ZFC (for more information on Q-sets see the survey paper **[F1]**).

A *strong Q-set* is a Q-set all finite powers of which are Q-sets. It is known that the Pixley-Roy hyperspace of a metric separable space M is a normal nonmetrizable Moore space if and only if M is an uncountable strong Q-set **[PT]**, **[R]**.

W. G. Fleissner proved that the square of a Q-set in general does not have to be a Q-set **[F2]**. We prove that the existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set.[1] We also formulate some other statements equivalent to the existence of uncountable Q-sets.

LEMMA 1. *If $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence of separable metrics σ_n on X then there exists a separable metric σ on X which is stronger than any of the metrics σ_n.*

PROOF. Consider the diagonal of $\prod_{n=1}^{\infty}(X, \sigma_n)$. □

LEMMA 2 (cf. **[BBM**, THEOREM 3]). *Let A be a subset of $X \times Y$ and ρ a separable metric on X. All horizontal sections of A (i.e. sets $A_y = \{x \in X: (x, y) \in A\}$ for $y \in Y$) are F_σ subsets of (X, ρ) if and only if there exists a separable metric σ on Y such that A is an F_σ subset of $(X, \rho) \times (Y, \sigma)$.*

PROOF. The "if" implication is obvious. Suppose that all horizontal sections A_y of A are F_σ subsets of (X, ρ). For every $y \in Y$ let $X \setminus A_y = \bigcap_{n=1}^{\infty} G_{y,n}$, where sets $G_{y,n}$ are open in (X, ρ) and let $\{B_m\}_{m=1}^{\infty}$ be a base for (X, ρ). Put $C_{n,m} = \{y \in Y: B_m \subset G_{y,n}\}$.

One easily verifies that

$$(X \times Y) \setminus A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (B_m \times C_{n,m}).$$

It suffices to find a separable metric σ on Y in which all sets $C_{n,m}$ are open (cf. Lemma 1). □

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[1]This answers a question brought to my attention by F. D. Tall.

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Let us denote by R the real line or any other set of cardinality continuum. As usual $\omega_1 = \{\alpha: \alpha < \omega_1\}$.

THEOREM. *The following conditions are equivalent*:

(i) *there exists an uncountable Q-set*;

(ii) *there exists an uncountable strong Q-set*;

(iii) $2^{\omega_1} = 2^{\omega}$ *and for every subset A of $R \times \omega_1$ there exist separable metrics ρ on R and σ on ω_1 such that A is an F_σ subset of $(R, \rho) \times (\omega_1, \sigma)$*;

(iv) $2^{\omega_1} = 2^{\omega}$ *and for every family $\mathcal{Q}$ of ω_1 subsets of R there exists a separable metric ρ on R such that all members of $\mathcal{Q}$ are F_σ subsets of (R, ρ)*;

(v) $2^{\omega_1} = 2^{\omega}$ *and for every $n < \omega$ and every subset A of $R \times \omega_1^n$ there exist separable metrics ρ on R and σ on ω_1 such that A is an F_σ subset of $(R, \rho) \times (\omega_1, \sigma)^n$*;

(vi) $2^{\omega_1} = 2^{\omega}$ *and for every $n < \omega$ and every family $\mathcal{Q}$ of ω_1 subsets of $R \times \omega_1^n$ there exist separable metrics ρ on R and σ on ω_1 such that all members of $\mathcal{Q}$ are F_σ subsets of $(R, \rho) \times (\omega_1, \sigma)^n$.*

REMARK 1. W. G. Fleissner proved that there exist models of set theory in which there exist Q-sets of cardinality ω_2, but in which there are no strong Q-sets of cardinality ω_2 **[F2]**. This implies that conditions (i) and (ii) in the above theorem are no longer equivalent if one assumes that the considered Q-sets are of cardinality ω_2. Similarly, conditions (iii) and (v) and conditions (iv) and (vi) are not equivalent if one replaces ω_1 by ω_2. □

REMARK 2. Conditions (i)–(vi) above are also equivalent to the following propositions (for more information, see **[P]**):

(vii) R^{ω_1} is a continuous image of a separable first countable space (here R carries its usual topology);

(viii) every space of cardinality (or weight) ω_1 can be embedded into a sequentially separable space. □

PROOF OF THE THEOREM. (i) $\rightarrow$ (iii). Let σ be a separable metric on ω_1 such that (ω_1, σ) is a Q-set. From Lemma 2 it follows that there exists a separable metric ρ on R such that A is an F_σ subset of $(R, \rho) \times (\omega_1, \sigma)$.

(iii) $\rightarrow$ (iv). Let $\mathcal{Q} = \{A_\alpha: \alpha < \omega_1\}$. It suffices to put $A = \bigcup \{A_\alpha \times \{\alpha\}: \alpha < \omega_1\} \subset R \times \omega_1$ and apply (iii).

(iv) $\rightarrow$ (v). For the sake of simplicity, we shall prove (v) only in the case of $n = 2$. The general case can be similarly proved by induction.

Let A be a subset of $R \times \omega_1 \times \omega_1$ and put $B = \{(r, \alpha, \beta) \in A: \alpha \leqslant \beta\}$ and $C = \{(r, \alpha, \beta) \in A: \alpha \geqslant \beta\}$. From the symmetry of assumptions and the equality $A = B \cup C$ we infer that without loss of generality we can assume that $A = B$. For every $\beta \in \omega_1$ put $A_\beta = \{(r, \alpha) \in R \times \omega_1: (r, \alpha, \beta) \in A\}$. From Lemma 2 it follows that it suffices to show that there exist separable metrics ρ on R and σ on ω_1 such that all sets A_β are F_σ subsets of $(R, \rho) \times (\omega_1, \sigma)$ for $\beta \in \omega_1$. Since $A_\beta \subset R \times (\beta + 1)$, for every $\beta < \omega_1$, it suffices to show that there exists a separable metric ρ on R such that all sets $A_{\beta\alpha} = \{r \in R: (r, \alpha) \in A_\beta\}$ are F_σ subsets of (R, ρ) for $\beta < \omega_1$ and $\alpha \leqslant \beta$, but this is a consequence of (iv).

(v) $\rightarrow$ (vi). Let $\mathcal{Q} = \{A_\alpha: \alpha < \omega_1\}$. It suffices to put $A = \bigcup \{A_\alpha \times \{\alpha\}: \alpha < \omega_1\} \subset R \times \omega_1^{n+1}$ and apply (v).

(vi) → (ii). It is enough to prove that for every $n < \omega$ there exists a separable metric σ_n on ω_1 such that $(\omega_1, \sigma_n)^n$ is a Q-set, because then, by Lemma 1, there would exist a σ which is stronger than any of the metrics σ_n and clearly (ω_1, σ) is a strong Q-set.

Let $n < \omega$ and let $\{A_r: r \in R\}$ be the family of all subsets of ω_1^n. Put $A = \bigcup \{\{r\} \times A_r: r \in R\} \subset R \times \omega_1^n$. By (vi) there exist separable metrics ρ on R and σ on ω_1 such that A is an F_σ subset of $(R, \rho) \times (\omega_1, \sigma)^n$. Clearly, $(\omega_1, \sigma)^n$ is a Q-set. □

Remark 3. R. Pol suggested a different proof of the equivalence of (i) and (ii) based on the fact that ω_1^2 is a union of countably many graphs and inverse graphs. □

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