THE EXISTENCE OF Q-SETS IS EQUIVALENT TO THE EXISTENCE OF STRONG Q-SETS

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Abstract. In this note we prove that the existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set, i.e. a Q-set all finite powers of which are Q-sets.

A Q-set is a separable metric space all subsets of which are Fσ sets. It is well known that the existence of an uncountable Q-set is equivalent to the existence of a normal separable nonmetrizable Moore space and is undecidable in ZFC (for more information on Q-sets see the survey paper [F1]).

A strong Q-set is a Q-set all finite powers of which are Q-sets. It is known that the Pixley-Roy hyperspace of a metric separable space M is a normal nonmetrizable Moore space if and only if M is an uncountable strong Q-set [PT], [R].

W. G. Fleissner proved that the square of a Q-set in general does not have to be a Q-set [F2]. We prove that the existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set. We also formulate some other statements equivalent to the existence of uncountable Q-sets.

Lemma 1. If \( \{ \sigma_n \}_{n=1}^{\infty} \) is a sequence of separable metrics \( \sigma_n \) on \( X \) then there exists a separable metric \( \sigma \) on \( X \) which is stronger than any of the metrics \( \sigma_n \).

Proof. Consider the diagonal of \( \prod_{n=1}^{\infty} (X, \sigma_n) \).

Lemma 2 (cf. [BBM, Theorem 3]). Let \( A \) be a subset of \( X \times Y \) and \( \rho \) a separable metric on \( X \). All horizontal sections of \( A \) (i.e. sets \( A_y = \{ x \in X : (x, y) \in A \} \) for \( y \in Y \)) are \( F_\sigma \) subsets of \( (X, \rho) \) if and only if there exists a separable metric \( \sigma \) on \( Y \) such that \( A \) is an \( F_\sigma \) subset of \( (X, \rho) \times (Y, \sigma) \).

Proof. The "if" implication is obvious. Suppose that all horizontal sections \( A_y \) of \( A \) are \( F_\sigma \) subsets of \( (X, \rho) \). For every \( y \in Y \) let \( X \setminus A_y = \bigcap_{n=1}^{\infty} G_{y,n} \), where sets \( G_{y,n} \) are open in \( (X, \rho) \) and let \( \{ B_m \}_{m=1}^{\infty} \) be a base for \( (X, \rho) \). Put \( C_{n,m} = \{ y \in Y : B_m \subseteq G_{y,n} \} \).

One easily verifies that

\[
(X \times Y) \setminus A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (B_m \times C_{n,m}).
\]

It suffices to find a separable metric \( \sigma \) on \( Y \) in which all sets \( C_{n,m} \) are open (cf. Lemma 1).

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Let us denote by $T$ the real line or any other set of cardinality continuum. As usual $\omega_1 = \{\alpha: \alpha < \omega_1\}$.

**Theorem.** The following conditions are equivalent:

(i) there exists an uncountable $Q$-set;

(ii) there exists an uncountable strong $Q$-set;

(iii) $2^{\omega_1} = 2^\omega$ and for every subset $A$ of $R \times \omega_1$ there exist separable metrics $\rho$ on $R$ and $\sigma$ on $\omega_1$ such that $A$ is an $F_\sigma$ subset of $(R, \rho) \times (\omega_1, \sigma)$;

(iv) $2^{\omega_1} = 2^\omega$ and for every family $\mathcal{Q}$ of $\omega_1$ subsets of $R$ there exists a separable metric $\rho$ on $R$ such that all members of $\mathcal{Q}$ are $F_\sigma$ subsets of $(R, \rho)$;

(v) $2^{\omega_1} = 2^\omega$ and for every $n < \omega$ and every subset $A$ of $R \times \omega_1^n$ there exist separable metrics $\rho$ on $R$ and $\sigma$ on $\omega_1$ such that $A$ is an $F_\sigma$ subset of $(R, \rho) \times (\omega_1, \sigma)^n$;

(vi) $2^{\omega_1} = 2^\omega$ and for every $n < \omega$ and every family $\mathcal{Q}$ of $\omega_1$ subsets of $R \times \omega_1^n$ there exist separable metrics $\rho$ on $R$ and $\sigma$ on $\omega_1$ such that all members of $\mathcal{Q}$ are $F_\sigma$ subsets of $(R, \rho) \times (\omega_1, \sigma)^n$.

**Remark 1.** W. G. Fleissner proved that there exist models of set theory in which there exist $Q$-sets of cardinality $\omega_2$, but in which there are no strong $Q$-sets of cardinality $\omega_2$ [F2]. This implies that conditions (i) and (ii) in the above theorem are no longer equivalent if one assumes that the considered $Q$-sets are of cardinality $\omega_2$. Similarly, conditions (iii) and (v) and conditions (iv) and (vi) are not equivalent if one replaces $\omega_1$ by $\omega_2$. □

**Remark 2.** Conditions (i)–(vi) above are also equivalent to the following propositions (for more information, see [P]):

(vii) $R^{\omega_1}$ is a continuous image of a separable first countable space (here $R$ carries its usual topology);

(viii) every space of cardinality (or weight) $\omega_1$ can be embedded into a sequentially separable space. □

**Proof of the Theorem.** (i) $\rightarrow$ (iii). Let $\sigma$ be a separable metric on $\omega_1$ such that $(\omega_1, \sigma)$ is a $Q$-set. From Lemma 2 it follows that there exists a separable metric $\rho$ on $R$ such that $A$ is an $F_\sigma$ subset of $(R, \rho) \times (\omega_1, \sigma)$.

(iii) $\rightarrow$ (iv). Let $\mathcal{Q} = \{A_\alpha: \alpha < \omega_1\}$. It suffices to put $A = \bigcup \{A_\alpha \times \{\alpha\}: \alpha < \omega_1\} \subset R \times \omega_1$ and apply (iii).

(iv) $\rightarrow$ (v). For the sake of simplicity, we shall prove (v) only in the case of $n = 2$. The general case can be similarly proved by induction.

Let $A$ be a subset of $R \times \omega_1 \times \omega_1$ and put $B = \{(r, \alpha, \beta) \in A: \alpha < \beta\}$ and $C = \{(r, \alpha, \beta) \in A: \alpha > \beta\}$. From the symmetry of assumptions and the equality $A = B \cup C$ we infer that without loss of generality we can assume that $A = B$. For every $\beta \in \omega_1$ put $A_\beta = \{(r, \alpha) \in R \times \omega_1: (r, \alpha, \beta) \in A\}$. From Lemma 2 it follows that it suffices to show that there exist separable metrics $\rho$ on $R$ and $\sigma$ on $\omega_1$ such that all sets $A_\beta$ are $F_\sigma$ subsets of $(R, \rho) \times (\omega_1, \sigma)$ for $\beta \in \omega_1$. Since $A_\beta \subset R \times (\beta + 1)$, for every $\beta < \omega_1$, it suffices to show that there exists a separable metric $\rho$ on $R$ such that all sets $A_{\beta\alpha} = \{r \in R: (r, \alpha) \in A_\beta\}$ are $F_\sigma$ subsets of $(R, \rho)$ for $\beta < \omega_1$ and $\alpha < \beta$, but this is a consequence of (iv).

(v) $\rightarrow$ (vi). Let $\mathcal{Q} = \{A_\alpha: \alpha < \omega_1\}$. It suffices to put $A = \bigcup \{A_\alpha \times \{\alpha\}: \alpha < \omega_1\}$ and apply (v).
(vi) → (ii). It is enough to prove that for every $n < \omega$ there exists a separable metric $\sigma_n$ on $\omega_1$ such that $(\omega_1, \sigma_n)^n$ is a $Q$-set, because then, by Lemma 1, there would exist a $\sigma$ which is stronger than any of the metrics $\sigma_n$ and clearly $(\omega_1, \sigma)$ is a strong $Q$-set.

Let $n < \omega$ and let $\{ A_r : r \in R \}$ be the family of all subsets of $\omega_1^n$. Put $A = \bigcup \{ \{ r \} \times A_r : r \in R \} \subset R \times \omega_1^n$. By (vi) there exist separable metrics $\rho$ on $R$ and $\sigma$ on $\omega_1$ such that $A$ is an $F_\sigma$ subset of $(R, \rho) \times (\omega_1, \sigma)^n$. Clearly, $(\omega_1, \sigma)^n$ is a $Q$-set.

$\square$

Remark 3. R. Pol suggested a different proof of the equivalence of (i) and (ii) based on the fact that $\omega_1^2$ is a union of countably many graphs and inverse graphs.

$\square$

REFERENCES


