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DECOMPOSITION OF RECURSIVELY ENUMERABLE DEGREES

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ABSTRACT. It is shown that any nonzero recursively enumerable degree can be expressed as the join of two distinct such degrees having a greatest lower bound.

Let a nonrecursive r.e. set A be given. We shall show how to enumerate r.e. sets B^0, B^1 and C such that C is recursive in A,

$$B^0 \cup B^1 = A, \qquad B^0 \cap B^1 = \varnothing,$$
$$B^{1-i} \nleq_T B^i \oplus C \quad (i = 0, 1)$$

and

$$\deg C = \deg(B^0 \oplus C) \cap \deg(B^1 \oplus C).$$

One corollary of this result is that the so-called "nondiamond" theorem [**1**, Theorem 5] cannot be improved to read: if $\mathbf{b}_0$, $\mathbf{b}_1$ are r.e. degrees such that $\mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{0}'$ and $\mathbf{b}_0|\mathbf{b}_1$ then $\mathbf{b}_0$ and $\mathbf{b}_1$ have no greatest lower bound in the upper semilattice of r.e. degrees. The same conclusion has been reached by J. R. Shoenfield and R. I. Soare [**4**] who independently and at about the same time as the present author constructed r.e. degrees $\mathbf{b}_0$, $\mathbf{b}_1$ such that $\mathbf{b}_0|\mathbf{b}_1$, $\mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{0}'$, and $\mathbf{b}_0 \cap \mathbf{b}_1$ exists. Our construction combines the technique for constructing minimal pairs from Lachlan [**1**, Theorem 1] and Yates [**6**] with the method of Sacks's splitting theorem [**3**, Theorem 1]. Our interest in this topic was awakened by Soare's article [**5**] where he asks whether improvement of the nondiamond theorem is possible.

We first establish some notation. Let $\langle \Phi_i \colon i < \omega \rangle$ and $\langle (\Psi_i^0, \Psi_i^1) \colon i < \omega \rangle$ be standard enumerations of all partial recursive (p.r.) functionals and of all ordered pairs of p.r. functionals. We assume given an enumeration of A and simultaneous uniformly effective enumerations of the p.r. functionals Φ_i, Ψ_i^0, and Ψ_i^1. In describing the enumeration of B^0, B^1, and C we often use our notations for sets and functionals to denote current approximations to them. If we wish to specify the approximation from a stage t other than the current one we append "$[t]$" to an expression. Thus $\Phi_i(B^0)[t]$ denotes the finite function obtained by applying the finite functional $\Phi_i[t]$, defined by the axioms of Φ_i enumerated before stage t, to $B^0[t]$ the set of numbers enumerated in B^0 before stage t. We use the same notation for a set and its characteristic function.

In the construction below we shall be satisfying the following requirements:

N_j: If $\Psi_j^0(B^0 \oplus C)$, $\Psi_j^1(B^1 \oplus C)$ are the same total function then that total function is recursive in C,

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$$R_{e,i}\colon B^{1-i} \neq \Phi_e(B^i \oplus C).$$

The lexicographic ordering of $\omega \times 2$ is denoted by $<$ and the requirement $R_{e,i}$ will be given priority according to the position of its index (e, i) in this ordering.

For unary partial functions F, G we let

$$l(F, G) = \sup\{n\colon (\forall i < n)(F(i), G(i) \text{ are both defined and equal})\}$$

and at any stage we set

$$l(i) = l(\Psi_i^0(B^0 \oplus C), \Psi_i^1(B^1 \oplus C)), \qquad l_e^i = l(B^{1-i}, \Phi_e(B^i \oplus C)),$$

and k_e^i equal the least number preserving $\Phi_e(B^i \oplus C)$ on l_e^i. We are free to assume that $k_e^i[s+1] \geqslant k_e^i[s]$ whenever $l_e^i[s+1] > l_e^i[s]$. This will be convenient in the proof of (5) below. For each stage s we define $\beta \in {}^{\omega}2$ by induction:

$$\beta(n) = 0 \leftrightarrow (\forall t < s)\big[(\beta \upharpoonright n)[t] = \beta \upharpoonright n \rightarrow l(n)[t] < l(n)\big].$$

The sequence β is used in the satisfaction of the requirements N_j. The technique for handling these requirements is similar to that employed in the version of the minimal pair construction presented in [2]. The idea is as follows. Let $<$ denote the lexicographic ordering of ${}^{<\omega}2$. Fix j and without loss suppose that $\Psi_j^0(B^0 \oplus C)$ and $\Psi_j^1(B^1 \oplus C)$ are the same total function. Let γ be least in ${}^{(j+1)}2$ such that $\gamma \subset \beta[s]$ for infinitely many s. Note that $\gamma(j) = 0$. The crucial stages for the satisfaction of N_j are those at which $\gamma \subset \beta$. Call them *γ-stages*. It does not matter that γ is unknown in the course of the construction, because we pursue the appropriate strategy for each possible value of γ giving priority to lesser values. We ignore the Sacks requirements $R_{e,i}$ with $e \leqslant j$, which are those accorded higher priority than N_j because they will only affect a finite number of stages. Likewise we ignore the finite number of stages with $\beta < \gamma$. The key apparatus of our construction is a strictly increasing sequence of markers $K(0), K(1), \ldots,$ an initial segment of which are defined in any particular stage, and each of which is eventually fixed. With each marker $K(m)$ will be associated $F(m)$ a number not yet in C which is defined when $K(m)$ is and which becomes fixed with $K(m)$. For each m, $K(m)$ will eventually be reset in a γ-stage and will be reset for the last time in a γ-stage. Let t be a γ-stage in which $K(m)$ is reset and let

$$E = \big(\Psi_j^0(B^0 \oplus C) \upharpoonright l(j)\big)[t] = \big(\Psi_j^1(B^1 \oplus C) \upharpoonright l(j)\big)[t].$$

At stage t we implicitly make a commitment to ensure that for all $s > t$ either

$$E \subset \big(\Psi_j^0(B^0 \oplus C) \upharpoonright l(j)[t]\big)[s]$$

or

$$E \subset \big(\Psi_j^1(B^1 \oplus C) \upharpoonright l(j)[t]\big)[s]$$

or

$$(\exists i < m)(F(i)[t] \in C[s] - C[t]).$$

This allows us to argue at the end of the construction that N_j will be satisfied.

The construction. We require auxiliary partial functions K, I, E, L, F and α. Initially, all these functions have empty domain and at the end of each stage they have as common domain some $n \in \omega$ and values in ω except for α whose values lie in ${}^{<\omega}2$. The meanings of these functions are as follows. For each $m \in \operatorname{dom} K$ we aim to attack the requirement with index $(E(m), I(m))$ by preserving $B^{I(m)}$ and C on $K(m)$. When $K(m)$ is set, $L(m)$ is the length of agreement of $B^{1-I(m)}$ and $\Phi_{E(m)}(B^{I(m)} \oplus C)$, and $\alpha(m)$ is $B \restriction E(m)$. We will choose $K(m)$ large enough to preserve $\Psi_j^{I(m)}(B^{I(m)} \oplus C)$ on $l(j)$ for each $j < E(m)$ such that $\beta(j) = 0$ thereby helping to meet requirements N_j. Finally, $F(m)$ is a number to be enumerated in C if at some later stage a number $\geqslant K(m-1)$ and $< K(m)$ is enumerated in A. Enumerating $F(m)$ in C assists in the satisfaction of the requirements N_j because it permits us to make a new prediction for the values of $\Psi_j^0(B^0 \oplus C) = \Psi_j^1(B^1 \oplus C)$ whenever necessary. This feature is what distinguishes the satisfaction of the requirements N_j from that of the corresponding requirements in the construction of a minimal pair where $\deg(C) = \mathbf{0}$.

The pair (e, i) is said to *require attention at stage s through m* if one of the following three possibilities holds.

$$m \in \operatorname{dom} K \,\&\, (e, i) = (E(m), I(m)) \,\&\, \beta \restriction e < \alpha(m) \restriction e; \tag{1}$$

$$\begin{aligned} &m \in \operatorname{dom} K \,\&\, (e, i) \leqslant (E(m), I(m)) \,\&\, \beta \restriction e \leqslant \alpha(m) \restriction e \\ &\& \max(k_e^i, l_e^i) > K(m-1) \\ &\& \left\{(e, i) = (E(m), I(m)) \to \left(\beta \restriction e = \alpha(m) \restriction e \,\&\, l_e^i > L(m)\right)\right\}; \end{aligned} \tag{2}$$

or

$$m = \operatorname{dom} K \,\&\, \max(k_e^i, l_e^i) > K(m-1). \tag{3}$$

By convention $K(-1) = 0$.

Stage s will have two parts. In part one we let (e, i) be the least pair requiring attention and we let m be the least number through which (e, i) requires attention. Delete any values the auxiliary functions may have for arguments $\geqslant m$. Set $E(m) = e$, $I(m) = i$, and $\alpha(m) = \beta \restriction e$. If (1) holds set $L(m) = L(m)[s]$ and $K(m)$ equal to the least number such that

$$K(m) \geqslant K(m)[s] \,\&\, (\forall j < e)\big(\beta(j) = 0 \to K(m) \text{ preserves } \Psi_j^i(B^i \oplus C) \text{ on } l(j)\big).$$

Otherwise set $L(m) = l_e^i$ and $K(m)$ equal to the least number such that

$$\begin{aligned} K(m) \geqslant \max(k_e^i, l_e^i) \,\&\, \big(m \in \operatorname{dom} K[s] \to K(m) \geqslant K(m)[s]\big) \\ \& (\forall j < e)\big(\beta(j) = 0 \to K(m) \text{ preserves } \Psi_j^i(B^i \oplus C) \text{ on } l(j)\big). \end{aligned}$$

In either case set $F(m)$ equal to the first unused number $\geqslant K(m)$. Part one is vacuous if there is no pair requiring attention.

In part two the next member, say n, is enumerated in A. If $n \geqslant \operatorname{rng} K$ enumerate n in B^0. Otherwise let m be the least number such that $n < K(m)$. Enumerate n in $B^{1-I(m)}$, $F(m)$ in C, and then set $F(m)$ equal the first unused number $\geqslant K(m)$. Delete all values of the auxiliary functions for arguments $> m$. This completes the construction.

We shall now show that at the end of any stage

$$m \in \operatorname{dom} K \to K(m) > L(m); \tag{4}$$

and

$$\begin{aligned} m_0 < m_1 \in \operatorname{dom} K \to \big[& K(m_0) < K(m_1) \\ & .\&.\ (E(m_0), I(m_0)) < (E(m_1), I(m_1)) \to \alpha(m_0) \restriction E(m_0) < \alpha(m_1) \restriction E(m_1) \\ & .\&.\ (E(m_1), I(m_1)) < (E(m_0), I(m_0)) \to \alpha(m_0) \restriction E(m_1) < \alpha(m_1) \restriction E(m_1) \big]. \end{aligned} \tag{5}$$

Also for stage s we have

$$\begin{aligned} & [m_0 < m_1 \in \operatorname{dom} K \,\&\, (e, i) \text{ receives attention through } m_0] \\ & \qquad \to [(e, i) < (E(m_1), I(m_1)) \,\&\, \beta \restriction e < \alpha(m_1) \restriction e]. \end{aligned} \tag{6}$$

The proofs of (4), (5), and (6) are by induction on the stage. We have (4) immediately from the way $K(m)$ and $L(m)$ are defined in stage s.

For proof by contradiction suppose that (5) fails for the first time at the end of stage s through m_0, $m_1 \in \operatorname{dom} K$ where $m_0 < m_1$. Then $(E(m_1), I(m_1))$ receives attention in stage s through m_1 and certainly $K(m_1)[s+1] > K(m_1)[s]$ if $m_1 \in \operatorname{dom} K[s]$. If $m_1 \notin \operatorname{dom} K[s]$ then in stage s we have

$$K(m_1) > \max(k_e^i, l_e^i) > K(m_1 - 1)[s]$$

whence $K(m_1)[s+1] > K(m_0)[s+1]$. Suppose $(E(m_0), I(m_0)) < (E(m_1), I(m_1))$ then $\alpha(m_0) \restriction E(m_0) < \alpha(m_1) \restriction E(m_0)$ otherwise $(E(m_0), I(m_0))$ would require attention at stage s through m_0 by (1). Suppose $(E(m_1), I(m_1)) < (E(m_0), I(m_0))$ and $\alpha(m_1) \restriction E(m_1) < \alpha(m_0) \restriction E(m_1)$ then, letting $e = E(m_1)$, $i = I(m_1)$ for the rest of the argument, in stage s we have

$$\max(k_e^i, l_e^i) > K(m_0 - 1) \tag{7}$$

and

$$(e, i) = (E(m_0), I(m_0)). \to .\ \beta \restriction e = \alpha(m_0) \restriction e \,\&\, l_e^i > L(m_0). \tag{8}$$

The reason for this is that these relationships certainly hold after the first part of stage t where t is the greatest number $< s$ such that $K(m_1)[t]$ is undefined. At stages $> t$ and $< s$ any number enumerated in $B^0 \cup B^1$ is $> K(m_1 - 1)$ whence

$$k_e^i[s] < k_e^i[t] \vee l_e^i[s] < l_e^i[t]. \to .\ l_e^i[t] > K(m_1 - 1)[t] \,\&\, l_e^i[s] > K(m_1 - 1)[s].$$

Since $K(m_1 - 1) > K(m_0)$ and $K(m_0) > L(m_0)$, $l_e^i[s] > K(m_1 - 1)[s]$ implies (7) and (8). This confirms the claim that (7) and (8) hold in stage s. Therefore, (e, i) requires attention through m_0 in stage s which is the desired contradiction. This completes the proof of (5).

For (6) suppose that $m_0 < m_1 \in \operatorname{dom} K[s]$ and (e, i) receives attention through m_0 at stage s. Then $(e, i) < (E(m_0), I(m_0))$ and $\beta \restriction e < \alpha(m_0) \restriction e$ from (1) and (2). For a contradiction argument let $(E(m_1), I(m_1)) < (e, i)$. Then $m_0 < m_1$ and $(E(m_1), I(m_1)) < (E(m_0), I(m_0))$ whence by (5) $\alpha(m_0) \restriction E(m_1) < \alpha(m_1) \restriction E(m_1)$. Since $\beta \restriction e < \alpha(m_0) \restriction e$ and $m_0 < m_1$ we have $\beta \restriction E(m_1) < \alpha(m_1) \restriction E(m_1)$. Thus

$(E(m_1), I(m_1))$ requires attention through m_1 at stage s which contradicts $(E(m_0), I(m_0))$ receiving attention. Therefore, $(e, i) < (E(m_1), I(m_1))$. As noted $\beta \upharpoonright e < \alpha(m_0) \upharpoonright e$ and by (5) $\alpha(m_0) \upharpoonright e < \alpha(m_1) \upharpoonright e$ since $e < E(m_0), E(m_1)$. Therefore, $(e, i) < \alpha(m_1) \upharpoonright e$ which completes the proof.

Correctness of the construction. We first show that in the limit each of the auxiliary functions is total. For proof by contradiction let m be the least number such that either $K(m)$ is eventually never defined or is reset possibly to the same value infinitely often. Suppose $K(m)$ is eventually never defined then B^1 is finite whence there exists e such that $B^1 = \Phi_e(B^0 \oplus C)$. Since $(e, 0)$ requires attention infinitely often and l_e^0 tends to infinity we have a contradiction. Thus $K(m)$ is reset infinitely often. The value of $K(m)$ is deleted at most finitely often in part two of a stage. Hence eventually $K(m)$ is always defined at the end of a stage. Once $K(m)$ is always defined $(E(m), I(m))$ and $\alpha(m)$ are nonincreasing, whence $E(m)$, $I(m)$ and $\alpha(m)$ are eventually fixed. From that point on when $K(m)$ is reset it means that (2) holds with $e = E(m)$ and $i = I(m)$, and in particular that $l_{E(m)}^{I(m)}$ has increased since the last time $K(m)$ was set. It follows that $K(m)$ and $L(m)$ increase without bound. Since $K(m-1)$ becomes fixed, eventually any number $< K(m)$ entering A is enumerated in $B^{1-I(m)}$ and no number $< K(m)$ is enumerated in $B^{I(m)}$ or C. In the limit $B^{I(m)}$ and C are recursive, $\deg B^{1-I(m)} = \deg A$, and $\Phi_{E(m)}(B^{I(m)} \oplus C) = B^{1-I(m)}$. This contradiction establishes our claim.

Suppose for contradiction that $B^{1-i} = \Phi_e(B^i \oplus C)$. Choose $\gamma \in {}^e2$ such that for infinitely many m, $\alpha(m) \upharpoonright e = \gamma$. Fix m such that $(e, i) < (E(m), I(m))$ and $\alpha(m) \upharpoonright e = \gamma$. This is possible because from (5) for each $f < \omega$ and $j < 2$ there are $< 2^f$ values of n such that $(E(n), I(n)) = (f, j)$. Now choose large s such that for some $n < m$, $K(n)$ is set or reset in stage s and such that at end of stage s, $(e, i) < (E(n), I(n))$ and $\alpha(n) \upharpoonright e = \gamma$. Thus $\beta \upharpoonright e[s] = \gamma$. Now $l_e^i[s] > K(m)$ since s is large whence (e, i) requires attention through m at stage s. Since $(E(n), I(n))$ actually receives attention at stage s we have the desired contradiction.

Notice that C is recursive in A because a number is enumerated in C only in response to a smaller number being enumerated in A. Next suppose

$$\Psi_q^0(B^0 \oplus C) = \Psi_q^1(B^1 \oplus C) = D$$

where D is total. To complete the proof of correctness we show that D is recursive in C. Fix $e > q$. Let γ be the least $\delta \in {}^e2$ such that $(\beta \upharpoonright e)[s] = \delta$ for infinitely many s. From (1) at the end of the construction for all m,

$$E(m) > e \rightarrow (\alpha(m) \upharpoonright e) < \gamma$$

and for all sufficiently large m we have equality on the right. From the definition of β, $\gamma(q) = 0$. Choose s_0 such that for all m if $E(m) < e$ or $\alpha(m) \upharpoonright e < \gamma$ in the limit then $K(m+1)[s]$ is defined for all $s > s_0$. Given an oracle for C we can compute $D(n)$ as follows. Seek s and m such that $s > s_0$, $(\beta \upharpoonright e)[s] = \gamma$, $l(q)[s] > n$, $K(m)$ is set or reset in stage s, $E(m) > e$, and none of $F(0)[s], F(1)[s], \ldots, F(m-1)[s]$ is ever enumerated in C. We claim that $D(n) = \Psi_q^0(B^0 \oplus C)(n)[s]$. Call (s', m') the *heir* of (s, m) if s' is the least stage $> s$ at which $K(p)$ is reset for some $p < m$, and $K(m')$ is reset at stage s'. Consider the sequence $(s, m), (s', m'), (s'', m''), \ldots$ where each term is the heir of the preceding one. There is a last member $(s^{(k)}, m^{(k)})$ because $m \geqslant m' \geqslant m'' \geqslant \cdots$ and the auxiliary functions all converge. Let $s^{(k+1)}$ denote ω. For all $j \leqslant k$, $K(m^{(j)})$ is not deleted in the second part of stage $s^{(j)}$ nor at any stage $> s^{(j)}$ and $< s^{(j+1)}$ because $F(h) = F(h)[s]$ for all $h < m^{(j)}$.

Let $e^{(j)}$, $i^{(j)}$ denote $E(m^{(j)})[s^{(j)} + 1]$, $I(m^{(j)})[s^{(j)} + 1]$ respectively for all $j \leqslant k$. From (6) for all $j < k$ we have

$$(e^{(j)}, i^{(j)}) \geqslant (e^{(j+1)}, i^{(j+1)})$$

and

$$(\alpha(m^{(j)}) \upharpoonright e^{(j+1)})[s^{(j)} + 1] \geqslant (\beta \upharpoonright e^{(j+1)})[s^{(j+1)}]$$
$$= (\alpha(m^{(j+1)}) \upharpoonright e^{(j+1)})[s^{(j+1)} + 1].$$

By choice of s_0, $E(m^{(k)})[s^{(k)} + 1] \geqslant e$ and $(\alpha(m^{(k)}) \upharpoonright e)[s^{(k)} + 1] \geqslant \gamma$. Hence $e^{(j)} \geqslant e$ for all $j \leqslant k$. Also, since $(\alpha(m) \upharpoonright e)[s + 1] = (\beta \upharpoonright e)[s] = \gamma$, for all $j \leqslant k$ we have

$$(\alpha(m^{(j)}) \upharpoonright e)[s^{(j)} + 1] = (\beta \upharpoonright e)[s^{(j)}] = \gamma.$$

Define $i(j) = I(m^{(j)})[s^{(j)} + 1]$. By induction on j, for all $j \leqslant k$: $l(q)[s^{(j)}] > n$,

$$\Psi_q^0(B^0 \oplus C)(n)[s^{(j)}] = \Psi_q^1(B^1 \oplus C)(n)[s^{(j)}],$$

in stage s_j $K(m^{(j)})$ is set to preserve $\Psi_q^{i(j)}(B^{i(j)} \oplus C)(n)$, no number $< K(m^{(j)})$ is enumerated in $B^{i(j)}$ or C at a stage $\geqslant s^{(j)}$ and $< s^{(j+1)}$, and $F(h)[s^{(j)}] = F(h)[s]$ for all $h < m^{(j)}$. It is now clear that for all $t \geqslant s$ there exists $i < 2$ such that

$$\Psi_q^i(B^i \oplus C)(n)[t] = \Psi_q^0(B^0 \oplus C)(n)[s].$$

This completes the proof of correctness.

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