HAVING A SMALL WEIGHT IS DETERMINED BY THE SMALL SUBSPACES

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Abstract. We show that for every cardinal \( \kappa > \omega \) and an arbitrary topological space \( X \) if we have \( w(Y) < \kappa \) whenever \( Y \subseteq X \) and \( |Y| < \kappa \) then \( w(X) < \kappa \) as well. M. G. Tkačenko proved this for \( T_3 \) spaces in [2]. We also prove an analogous statement for the \( \pi \)-weight if \( \kappa \) is regular.

The main aim of this paper is to prove the following result.

Theorem. Let \( X \) be an arbitrary topological space and \( \kappa > \omega \) a (regular) cardinal. If \( w(Y) < \kappa \) (\( \pi(Y) < \kappa \)) holds whenever \( Y \subseteq X \) and \( |Y| < \kappa \) then \( w(X) < \kappa \) (\( \pi(X) < \kappa \)).

In [2] M. G. Tkačenko proved this (with \( w \) only) for \( T_3 \) spaces and raised the question whether \( T_3 \) could be replaced by \( T_2 \). As we see, actually no separation axiom is needed.

We start to prove our theorem by establishing a lemma which might be of some interest in itself. We shall need the following piece of notation in stating it and also later. For an arbitrary set \( \mathcal{X} \), a family \( \mathcal{S} \) of subsets of \( \mathcal{X} \) and \( Y \subseteq \mathcal{X} \) we put
\[
\mathcal{S} \upharpoonright Y = \{ S \cap Y : S \in \mathcal{S} \},
\]
the trace of \( \mathcal{S} \) on \( Y \).

Lemma. Let \( X \) be an arbitrary topological space and \( \kappa > \omega \) be a regular cardinal. Moreover let \( (Y_\alpha : \alpha \in \kappa) \) be an increasing sequence of subspaces of \( X \) (i.e. \( Y_\alpha \subseteq Y_\beta \) if \( \alpha < \beta \)). If \( \mathcal{S} \) is a family of open subsets of \( X \) such that \( \mathcal{S} \upharpoonright Y_\alpha \) is a base (\( \pi \)-base) for \( Y_\alpha \) for each \( \alpha \in \kappa \), then \( \mathcal{S} \upharpoonright \bigcup \{ Y_\alpha : \alpha \in \kappa \} \) is also a base (\( \pi \)-base) for \( \bigcup \{ Y_\alpha : \alpha \in \kappa \} \) provided that \( X \) contains no left-separated subspace of cardinality \( \kappa \) (or equivalently, every subspace of \( X \) has a dense subset of cardinality less than \( \kappa \), cf. [1]).

Proof. We shall give the proof for the case of a base only, since that of the \( \pi \)-base is completely analogous. Suppose, on the contrary, that \( \mathcal{S} \upharpoonright Y \) is not a base for \( Y \). Then there is a point \( p \in Y \) and a neighbourhood \( U \) of \( p \) such that if \( p \in G \in \mathcal{S} \) then \( G \cap Y \nsubseteq G \cap U \). Now we select by transfinite induction members \( G_\alpha \in \mathcal{S} \) and points \( q_\alpha \in Y \cap G_\alpha \setminus U \) as follows. Assume \( \mu < \kappa \) and \( G_\mu \), \( q_\mu \) have already been selected for \( \nu < \mu \). Since \( \kappa \) is regular we can find an ordinal \( \alpha_\mu < \kappa \) such that \( p \in Y_{\alpha_\mu} \) and \( q_\nu \in Y_{\alpha_\mu} \) for every \( \nu < \mu \). By our assumption there is a \( G_\mu \in \mathcal{S} \) such that \( p \in G_\mu \cap Y_{\alpha_\mu} \subseteq U \). Then we can pick a point \( q_\mu \in G_\mu \cap Y \setminus U \). It is clear from our construction that if \( \nu < \mu < \kappa \) then \( q_\nu \notin G_\mu \); consequently the
sequence \( \{ q_\alpha \colon \alpha \in \kappa \} \) is left-separated, a contradiction.

**Proof of the Theorem.** Again we restrict ourselves to the case of the weight function \( w \), as that of \( \pi \) is done similarly. Moreover, we first assume that \( \kappa \) is regular. Our proof in this case is indirect, i.e. we assume \( w(X) > \kappa \).

Then we define by transfinite induction subspaces \( Y_\alpha \subseteq X \) and families of open sets \( \mathcal{B}_\alpha \) with \( |Y_\alpha| < \kappa \) and \( |\mathcal{B}_\alpha| < \kappa \) for \( \alpha < \kappa \) in the following way. Suppose that \( \alpha < \kappa \) and \( \mathcal{B}_\beta, \mathcal{B}_\beta \) have been defined for each \( \beta < \alpha \). If \( \alpha \) is limit (or 0) we put \( Y_\alpha = \bigcup \{ Y_\beta \colon \beta < \alpha \} \), and \( \mathcal{B}_\alpha \supseteq \bigcup \{ \mathcal{B}_\beta \colon \beta < \alpha \} \) is chosen in such a way that \( \mathcal{B}_\alpha \uparrow Y_\alpha \) is a base for \( Y_\alpha \) and \( |\mathcal{B}_\alpha| < \kappa \). This is possible because \( |Y_\alpha| < \kappa \) by the regularity of \( \kappa \). Now, if \( \alpha = \beta + 1 \), by our indirect assumption \( \mathcal{B}_\beta \) is not a base for \( X \), hence we can find a point \( p_\beta \in X \) and its neighbourhood \( U \) in such a way that no \( B \in \mathcal{B}_\beta \) satisfies \( p(\beta) \in B \subset U \). Let us put \( \mathcal{B}_\beta = \{ B \in \mathcal{B}_\beta : p(\beta) \in B \} \), then we can choose for each \( B \in \mathcal{B}_\beta \) a point \( q_\beta \in B \setminus U \). Finally, we put \( Y_\alpha = Y_\beta \cup \{ q_\beta \colon B \in \mathcal{B}_\beta \} \) and \( \mathcal{B}_\alpha \supseteq \mathcal{B}_\beta \) is chosen again so that \( \mathcal{B}_\alpha \uparrow Y_\alpha \) is a base for \( Y_\alpha \) and \( |\mathcal{B}_\alpha| < \kappa \). Let us note that then \( \mathcal{B}_\beta \uparrow Y_{\beta+1} \) is not a base for \( Y_{\beta+1} \). Having completed the induction we put

\[
Y = \bigcup \{ Y_\alpha : \alpha \in \kappa \} \quad \text{and} \quad \mathcal{B} = \bigcup \{ \mathcal{B}_\alpha : \alpha \in \kappa \}.
\]

Now, observe that \( w(Z) < \kappa \) for each \( Z \subseteq X \), \( |Z| < \kappa \) implies \( d(Z) < \kappa \) for each such \( Z \); consequently the conditions of our lemma are satisfied with the sequence of subspaces \( \langle Y_\alpha : \alpha \in \kappa \rangle \) and the open family \( \mathcal{B} \). Therefore \( \mathcal{B} \uparrow Y \) forms a base for \( Y \). But \( |Y| < \kappa \); hence by our assumption \( w(Y) < \kappa \) as well. Consequently, as is well known, we can select a subfamily \( C \subseteq \mathcal{B} \) with \( |C| = w(Y) < \kappa \) such that \( C \uparrow Y \) is already a base for \( Y \). Since \( \kappa \) is regular we must have then an \( \alpha < \kappa \) with \( C \subseteq \mathcal{B}_\alpha \). But, by our construction, \( \mathcal{B}_\alpha \uparrow Y_{\alpha+1} \) is not a base for \( Y_{\alpha+1} \) and, a fortiori, \( \mathcal{B}_\alpha \uparrow Y \) is not a base for \( Y \), a contradiction. This completes the proof for \( \kappa \) regular.

Now let us consider the case in which \( \kappa \) is singular. We claim that then there is a cardinal \( \lambda < \kappa \) such that actually \( w(Y) < \lambda \) holds whenever \( Y \subseteq X \) and \( |Y| < \kappa \). Assume, on the contrary, that no such \( \lambda \) exists. Then we can find for each cardinal \( \lambda < \kappa \) a subspace \( Y_\lambda \subseteq X \) with \( |Y_\lambda| < \kappa \) and \( w(Y_\lambda) > \lambda \). But putting \( Y = \bigcup \lambda < \kappa Y_\lambda \), we would have then \( |Y| < \kappa \) and \( w(Y) > \kappa \) (since \( \kappa \) is a limit cardinal), which is impossible.

Now take any regular \( \lambda < \kappa \) as in our claim. Then we can apply the first half of our proof to this \( \lambda \) to conclude that \( w(X) < \lambda < \kappa \).

The reader should notice that, since the \( \pi \)-weight is not monotone for subspaces, the second half of our proof (for \( \kappa \) singular) cannot be applied to it. Thus e.g. the following problem remains open.

**Problem.** Does there exist a topological space \( X \) such that \( \pi w(Y) < \aleph_\omega \) whenever \( Y \subseteq X \) and \( |Y| < \aleph_\omega \) but \( \pi w(X) > \aleph_\omega \)?

**References**


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