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HAVING A SMALL WEIGHT IS DETERMINED BY THE SMALL SUBSPACES

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ABSTRACT. We show that for every cardinal $\kappa > \omega$ and an arbitrary topological space X if we have $w(Y) < \kappa$ whenever $Y \subset X$ and $|Y| < \kappa$ then $w(X) < \kappa$ as well. M. G. Tkačenko proved this for T_3 spaces in [2]. We also prove an analogous statement for the π-weight if κ is regular.

The main aim of this paper is to prove the following result.

THEOREM. *Let X be an arbitrary topological space and $\kappa > \omega$ a (regular) cardinal. If $w(Y) < \kappa$ ($\pi(Y) < \kappa$) holds whenever $Y \subset X$ and $|Y| < \kappa$ then $w(X) < \kappa$ ($\pi(X) < \kappa$).*

In [2] M. G. Tkačenko proved this (with w only) for T_3 spaces and raised the question whether T_3 could be replaced by T_2. As we see, actually no separation axiom is needed.

We start to prove our theorem by establishing a lemma which might be of some interest in itself. We shall need the following piece of notation in stating it and also later. For an arbitrary set X, a family $\mathbb{S}$ of subsets of X and $Y \subset X$ we put $\mathbb{S} \upharpoonright Y = \{S \cap Y: S \in \mathbb{S}\}$, the trace of $\mathbb{S}$ on Y.

LEMMA. *Let X be an arbitrary topological space and $\kappa > \omega$ be a regular cardinal. Moreover let $(Y_\alpha: \alpha \in \kappa)$ be an increasing sequence of subspaces of X (i.e. $Y_\alpha \subset Y_\beta$ if $\alpha < \beta$). If $\mathcal{G}$ is a family of open subsets of X such that $G \upharpoonright Y_\alpha$ is a base (π-base) for Y_α for each $\alpha \in \kappa$, then $G \upharpoonright Y$ is also a base (π-base) for $Y = \bigcup \{Y_\alpha: \alpha \in \kappa\}$ provided that X contains no left-separated subspace of cardinality κ (or equivalently, every subspace of X has a dense subset of cardinality less than κ, cf.* [1]).

PROOF. We shall give the proof for the case of a base only, since that of the π-base is completely analogous. Suppose, on the contrary, that $\mathcal{G} \upharpoonright Y$ is not a base for Y. Then there is a point $p \in Y$ and a neighbourhood U of p such that if $p \in G \in \mathcal{G}$ then $G \cap Y \not\subset G \cap U$. Now we select by transfinite induction members $G_\nu \in \mathcal{G}$ and points $q_\nu \in Y \cap G_\nu \setminus U$ as follows. Assume $\mu < \kappa$ and G_ν, q_ν have already been selected for $\nu < \mu$. Since κ is regular we can find an ordinal $\alpha_\mu < \kappa$ such that $p \in Y_{\alpha_\mu}$ and $q_\nu \in Y_{\alpha_\mu}$ for every $\nu < \mu$. By our assumption there is a $G_\mu \in \mathcal{G}$ such that $p \in G_\mu \cap Y_{\alpha_\mu} \subset U$. Then we can pick a point $q_\mu \in G_\mu \cap Y \setminus U$. It is clear from our construction that if $\nu < \mu < \kappa$ then $q_\nu \notin G_\mu$; consequently the

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sequence $\{q_\nu: \nu \in \kappa\}$ is left-separated, a contradiction.

PROOF OF THE THEOREM. Again we restrict ourselves to the case of the weight function w, as that of π is done similarly. Moreover, we first assume that κ is regular. Our proof in this case is indirect, i.e. we assume $w(X) \geqslant \kappa$.

Then we define by transfinite induction subspaces $Y_\alpha \subset X$ and families of open sets $\mathfrak{B}_\alpha$ with $|Y_\alpha| < \kappa$ and $|\mathfrak{B}_\alpha| < \kappa$ for $\alpha < \kappa$ in the following way. Suppose that $\alpha < \kappa$ and Y_β, $\mathfrak{B}_\beta$ have been defined for each $\beta < \alpha$. If α is limit (or 0) we put $Y_\alpha = \bigcup \{Y_\beta: \beta < \alpha\}$, and $\mathfrak{B}_\alpha \supset \bigcup \{\mathfrak{B}_\beta: \beta < \alpha\}$ is chosen in such a way that $\mathfrak{B}_\alpha \restriction Y_\alpha$ is a base for Y_α and $|\mathfrak{B}_\alpha| < \kappa$. This is possible because $|Y_\alpha| < \kappa$ by the regularity of κ. Now, if $\alpha = \beta + 1$, by our indirect assumption $\mathfrak{B}_\beta$ is not a base for X, hence we can find a point $p^{(\beta)} \in X$ and its neighbourhood U in such a way that no $B \in \mathfrak{B}_\beta$ satisfies $p^{(\beta)} \in B \subset U$. Let us put $\mathfrak{B}_\beta^* = \{B \in \mathfrak{B}_\beta: p^{(\beta)} \in B\}$, then we can choose for each $B \in \mathfrak{B}_\beta^*$ a point $q_\beta \in B \setminus U$. Finally, we put $Y_\alpha = Y_\beta \cup \{p^{(\beta)}\} \cup \{q_\beta: B \in \mathfrak{B}_\beta^*\}$ and $\mathfrak{B}_\alpha \supset \mathfrak{B}_\beta$ is chosen again so that $\mathfrak{B}_\alpha \restriction Y_\alpha$ is a base for Y_α and $|\mathfrak{B}_\alpha| < \kappa$. Let us note that then $\mathfrak{B}_\beta \restriction Y_{\beta+1}$ is not a base for $Y_{\beta+1}$. Having completed the induction we put

$$Y = \bigcup \{Y_\alpha: \alpha \in \kappa\} \quad \text{and} \quad \mathfrak{B} = \bigcup \{\mathfrak{B}_\alpha: \alpha \in \kappa\}.$$

Now, observe that $w(Z) < \kappa$ for each $Z \subset X$, $|Z| \leqslant \kappa$ implies $d(Z) < \kappa$ for each such Z; consequently the conditions of our lemma are satisfied with the sequence of subspaces $\langle Y_\alpha: \alpha \in \kappa \rangle$ and the open family $\mathfrak{B}$. Therefore $\mathfrak{B} \restriction Y$ forms a base for Y. But $|Y| \leqslant \kappa$; hence by our assumption $w(Y) < \kappa$ as well. Consequently, as is well known, we can select a subfamily $\mathcal{C} \subset \mathfrak{B}$ with $|\mathcal{C}| = w(Y) < \kappa$ such that $\mathcal{C} \restriction Y$ is already a base for Y. Since κ is regular we must have then an $\alpha < \kappa$ with $\mathcal{C} \subset \mathfrak{B}_\alpha$. But, by our construction, $\mathfrak{B}_\alpha \restriction Y_{\alpha+1}$ is not a base for $Y_{\alpha+1}$ and, a fortiori, $\mathfrak{B}_\alpha \restriction Y$ is not a base for Y, a contradiction. This completes the proof for κ regular.

Now let us consider the case in which κ is singular. We claim that then there is a cardinal $\lambda < \kappa$ such that actually $w(Y) < \lambda$ holds whenever $Y \subset X$ and $|Y| \leqslant \kappa$. Assume, on the contrary, that no such λ exists. Then we can find for each cardinal $\lambda < \kappa$ a subspace $Y_\lambda \subset X$ with $|Y_\lambda| \leqslant \kappa$ and $w(Y_\lambda) \geqslant \lambda$. But putting $Y = \bigcup_{\lambda<\kappa} Y_\lambda$, we would have then $|Y| \leqslant \kappa$ and $w(Y) \geqslant \kappa$ (since κ is a limit cardinal), which is impossible.

Now take any regular $\lambda < \kappa$ as in our claim. Then we can apply the first half of our proof to this λ to conclude that $w(X) < \lambda < \kappa$.

The reader should notice that, since the π-weight is not monotone for subspaces, the second half of our proof (for κ singular) cannot be applied to it. Thus e.g. the following problem remains open.

PROBLEM. Does there exist a topological space X such that $\pi w(Y) < \aleph_\omega$ whenever $Y \subset X$ and $|Y| \leqslant \aleph_\omega$ but $\pi w(X) \geqslant \aleph_\omega$?

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