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K_2 MEASURES EXCISION FOR K_1

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ABSTRACT. Let B be a commutative ring, A a subring of B, and I an ideal of B contained in A. Excision holds if $K_1(A, I)$ and $K_1(B, I)$ are isomorphic. We show that the obstruction to excision holding is a subquotient of $K_2(B/I^2)$. We then use this obstruction to show that, if A and B are fixed, the excision problem for I has no bearing on the excision problem for ideals contained in I.

1. Introduction. This paper is concerned with computations in algebraic K-theory relating to the following situation:

Standing notation. Let A and B be two fixed commutative rings such that A is a subring of B. When I is an ideal of B contained in A, we will say that "excision holds for I" if the natural map $K_1(A, I) \to K_1(B, I)$ is an isomorphism. Otherwise, we will say that excision fails for I.

It is well known [**S**, p. 235] that $K_1(A, I) \to K_1(B, I)$ is a surjection; the kernel of this map will be denoted Φ_I. Note that Φ_I also depends on the rings A and B. The "excision problem for I" is to determine Φ_I and its image in $K_1(A)$. Thus excision holds if and only if $\Phi_I = 0$. Note that $K_1(A, I) = SK_1(A, I) \oplus \{\text{units in } 1 + I\}$, and that Φ_I is also the kernel of the surjection $SK_1(A, I) \to SK_1(B, I)$.

A reason to study the excision problem is that one would like to compute the K-theory of A, given the K-theory of B, A/I, and B/I. When excision holds, there is a Mayer-Vietoris sequence connecting K_2 and K_1. This is the special case $\Phi_I = 0$ of the following result, which may be proven by chasing the diagram on p. 55 of [**Mil**].

PROPOSITION 1.1. *There is an exact sequence*

$$K_2(B) \oplus K_2(A/I) \to K_2(B/I) \to K_1(A)/\,(\textit{image of } \Phi_I)$$
$$\to K_1(B) \oplus K_1(A/I) \to K_1(B/I).$$

Several authors have used the Mayer-Vietoris sequence (or variants) to compute K_1 and K_2 of rings found in algebraic geometry. Among these are [**DR**], [**DW**], [**GR1**], [**GR2**], [**K**], [**P**], [**R**]. Several authors have also used the Mayer-Vietoris sequence to compute K_i of the group ring $A = \mathbf{Z}[G]$ for various groups G. A nice summary of these computations may be found in [**St2**]. The Whitehead group $\mathrm{Wh}(G) = K_1(A)/\{\pm G\}$ is an important invariant of the group G.

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The first examples in which excision failed were reported by Swan in **[Sw]** and **[St1]**. In **[Sw]** it was proven that (for A, B commutative) Φ_I is generated by the Mennicke symbols $[{}_{1\ -bx}^{\ x}]$ with $x \in I$, $b \in B$. The Mennicke symbol $[{}_{1\ -bx}^{\ x}]$ is the element of $SK_1(A, I)$ represented by the matrix

$$\begin{pmatrix} 1 - bx & x \\ -b^2x & 1 + bx \end{pmatrix}.$$

Rules for manipulating Mennicke symbols may be found on p. 125 of **[Mil]** or on pp. 228–229 of **[Sw]**.

Later refinements **[GR2]**, **[V]** of Swan's work have shown that there is a surjection ε: $I/I^2 \otimes_B \Omega_{B/A} \to \Phi_I$ given by $\varepsilon(x \otimes db) = [{}_{1\ -bx}^{\ x}]$. We call ε the "Swan-Vorst map". Here $\Omega_{B/A}$ is the B-module of Kähler differentials of B over A. It is generated by symbols db, $b \in B$, with relations (1) $d(b_1 + b_2) = db_1 + db_2$, (2) $d(b_1b_2) = b_1db_2 + b_2db_1$ and (3) $da = 0$ for all $a \in A$.

The main results of this paper are that the sequence containing the Swan-Vorst map can be extended to a K_2 term, and that Φ_I is a subquotient of $K_2(B/I^2)$:

$$\Phi_I = \frac{\operatorname{Im} K_2(B/I^2, I/I^2) + \operatorname{Im} K_2(B)}{\operatorname{Im} K_2(A/I^2, I/I^2) + \operatorname{Im} K_2(B)}.$$

We obtain an exact sequence $K_2(B, I) \to I/I^2 \otimes \Omega_{B/A} \xrightarrow{\varepsilon} \Phi_I \to 0$ which may be rewritten as the exact sequence

$$\begin{aligned} K_2(B, I) \oplus K_2(A/I^2, I/I^2) &\to K_2(B/I^2, I/I^2) \to K_1(A, I) \\ &\to K_1(B, I) \oplus K_1(A/I^2, I/I^2) \to K_1(B/I^2, I/I^2) \ldots . \end{aligned}$$

The advantage of this formulation is that the image of $K_2(B/I^2, I/I^2)$ in $K_2(B/I^2)$ can frequently be computed using techniques recently developed in **[MS]**, **[ST]** and **[B]**. In §3 we will illustrate our point by providing counterexamples to the following questions, raised in **[GR2**, §10].

1. If excision holds (resp. fails) for I, does excision always hold or fail uniformly for ideals I^m when m is sufficiently large?

2. If excision holds (resp. fails) for I, does excision always hold or fail uniformly for sufficiently small ideals $J \subset I$?

In particular the following is a counterexample to question 1 (see §3 for proofs).

COUNTEREXAMPLE. Let k be a nonperfect field of characteristic $p \geqslant 3$, and let $B = k[t]$, $A = k[t^2, t^3] = k \oplus t^2B$. Then excision fails for the ideal $I = t^NB$ if and only if $N \equiv -1 \pmod p$. Consequently excision fails for I^m, $I = t^2B$, if and only if $M \equiv -\frac{1}{2} \pmod p$.

2. Excision and K_2. In this section we will identify the kernel of excision with a subquotient of K_2, and relate this to the Swan-Vorst map. We will also describe the cokernel of $\Phi_J \to \Phi_I$ for ideals $J \subset I$.

By $K_2(R, J)$ we mean Keune's relative term **[Keu**, p. 171], which is a quotient of

the "Stein relativization" described in [**Mil**] on p. 53. When J is a radical ideal of R, $K_2(R, J)$ is generated by symbols of the form $\langle x, r\rangle$, and $\langle r, x\rangle$ where $x \in J$ and $r \in R$ [**Keu**, p. 175]. The definition of the $\langle x, r\rangle$ and their properties are given on pp. 174–175 of [**Keu**].

For our purposes we will only need to know [**W**, Theorem 1.3] that when $J^2 = 0$ there is an exact sequence of abelian groups

$$J \otimes_R J \xrightarrow{\rho} K_2(R, J) \xrightarrow{\phi} J \otimes_R \Omega_{R/J} \to 0,$$

where $\rho(x \otimes y) = \langle x, y\rangle$ and $\phi\langle x, r\rangle = x \otimes dr$. From this we obtain the following simple result.

LEMMA 2.1. *Under our standing notation, with $I^2 \subseteq J \subset I$, the cokernel of $K_2(A/J, I/J) \to K_2(B/J, I/J)$ is isomorphic to $I/J \otimes_B \Omega_{B/A}$ under the map $\langle x, b\rangle \to x \otimes db$, where $x \in I/J$, $b \in B/J$.*

PROOF. Since $(I/J)^2 = 0$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
I/J \otimes_A I/J & \to & K_2(A/J, I/J) & \xrightarrow{\phi} & I/J \otimes_{A/I} \Omega_{A/I} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
I/J \otimes_B I/J & \to & K_2(B/J, I/J) & \xrightarrow{\phi} & I/J \otimes_{B/I} \Omega_{B/I} & \to & 0
\end{array}$$

A diagram chase shows that the cokernels of the middle and right vertical arrows are isomorphic (the left vertical map is onto). Since

$$I/J \otimes_{A/I} \Omega_{A/I} \simeq I/J \otimes_{B/I}(B/I \otimes_{A/I} \Omega_{A/I}),$$

the desired cokernel is

$$I/J \otimes_{B/I} \Omega_{B/I}/\mathrm{im}(B/I \otimes_{A/I} \Omega_{A/I}).$$

By Theorem 57 of [**Mat**, p. 186], $\Omega_{B/I}/\mathrm{im}(B/I \otimes_{A/I} \Omega_{A/I}) \simeq \Omega_{(B/I)/(A/I)}$. Moreover, $\Omega_{(B/I)/(A/I)} \simeq \Omega_{B/A} \otimes_B B/I \simeq \Omega_{B/A}$; this follows from [**Mat**, p. 186, Example 2] and the fact that $idb = idb + bdi = d(bi) = 0$ for $i \in I$. Therefore the cokernel is $I/J \otimes \Omega_{B/A}$ as claimed.

If we set $J = I^2$ in the lemma, we get the group $I/I^2 \otimes \Omega_{B/A}$. The natural thing to do now is to try to relate the groups in the lemma to the Swan-Vorst map. To do this, we compare the long exact ideal sequences of I and J. Note that our choice of relative K_2 is such that for any ideal $\mathfrak{a}$ in any ring Λ there is an exact sequence $K_3(\Lambda) \to K_3(\Lambda/\mathfrak{a}) \to K_2(\Lambda, \mathfrak{a}) \to K_2(\Lambda)$ [**Keu**, p. 159]. Thus if $\mathfrak{a} \subset \mathfrak{b}$ is a pair of ideals, we can extend the commutative diagram of interlocking sequences on p. 56 of [**Mil**] to the left and obtain the exactness of the sequence (*) on p. 56 of [**Mil**]. This shows that the rows are exact in the following commutative diagram (whose columns are also exact). Note that since I/J is nilpotent, we have $SK_1(A/J, I/J) = SK_1(B/J, I/J) = 0$ by [**Bass**, p. 469], whence the right-hand zeros in the diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Phi_J \arrow[r] \arrow[d] & \Phi_I \arrow[d] & \\
 & K_2(A/J, I/J) \arrow[r] \arrow[d, "\eta"] & SK_1(A, J) \arrow[r] \arrow[d] & SK_1(A, I) \arrow[r] \arrow[d] & 0 \\
K_2(B, I) \arrow[r] & K_2(B/J, I/J) \arrow[r, "\partial"] \arrow[d] & SK_1(B, J) \arrow[r] \arrow[d] & SK_1(B, I) \arrow[r] \arrow[d] & 0 \\
 & I/J \otimes_B \Omega_{B/A} \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{2.2}$$

A variant on the snake lemma gives a map from $I/J \otimes \Omega_{B/A}$ onto the cokernel of $\Phi_J \to \Phi_I$. This is the map we want.

PROPOSITION 2.3. *Under our standing notation, with $I^2 \subseteq J \subset I$, the map $I/J \otimes \Omega_{B/A} \to \operatorname{coker}(\Phi_J \to \Phi_I)$ given by* (2.2) *is the Swan-Vorst map up to sign.*

PROOF. Let $x \in I$, $b \in B$ have images $\bar{x}$, $\bar{b} \in B/J$. To compute the image of $\bar{x} \otimes d\bar{b}$ in Φ_I, we first have to compute $\partial(\langle \bar{x}, \bar{b} \rangle)$ in $SK_1(B, J)$. Now $\langle \bar{x}, \bar{b} \rangle \in K_2(B/J) \subset St(B/J)$ is the image of the following element of $St(B)$:

$$X_{-\alpha}(-b(1-bx))X_\alpha(x)X_{-\alpha}(b)X_\alpha(-x)X_\alpha(1)X_{-\alpha}(-1)X_\alpha(1)$$
$$\cdot X_\alpha(-(1+bx))X_{-\alpha}(1-bx)X_\alpha(-(1+bx)).$$

Mapping this element to $\mathrm{Gl}(B)$ with $\alpha = 12$, $-\alpha = 21$, and multiplying the first four and last six matrices together yields

$$\begin{pmatrix} 1+bx & -bx^2 \\ * & * \end{pmatrix}\begin{pmatrix} 1-bx & b^2x^2 \\ * & * \end{pmatrix}.$$

Each of these matrices actually lies in $\mathrm{Sl}(J) \subset \mathrm{Sl}(I)$. Thus in $SK_1(A, I)$ their product is given by the Mennicke symbols

$$\begin{bmatrix} -bx^2 \\ 1+bx \end{bmatrix}\begin{bmatrix} b^2x^2 \\ 1-bx \end{bmatrix} = \begin{bmatrix} -bx \\ 1+bx \end{bmatrix}\begin{bmatrix} x \\ 1+bx \end{bmatrix}\begin{bmatrix} bx \\ 1-bx \end{bmatrix}^2 = \begin{bmatrix} x \\ 1+bx \end{bmatrix}.$$

This element is just $\varepsilon(x \otimes db)^{-1}$, as claimed.

REMARK. If $J \subseteq I^2$, the case not covered by Proposition 2.3, it is known [**Sw**, p. 238] that $\Phi_J \to \Phi_I$ is zero. This may be seen from the fact that the map $J/J^2 \otimes \Omega_{B/A} \to I/I^2 \otimes \Omega_{B/A}$ is zero. In particular, if we take $J = I^2$ we obtain the map $-\varepsilon$ from $I/I^2 \otimes \Omega_{B/A}$ onto Φ_I.

Another chase on (2.2) with $J = I^2$ yields

COROLLARY 2.4. *The following two sequences are exact*:

$$K_2(B, I) \to I/I^2 \otimes_B \Omega_{B/A} \to SK_1(A, I) \to SK_1(B, I) \to 0,$$

$$K_2(B, I) \oplus K_2(A/I^2, I/I^2) \to K_2(B/I^2, I/I^2) \to SK_1(A, I) \to SK_1(B, I) \to 0.$$

The disadvantage of these sequences is that we do not know generators for $K_2(B, I)$ in any useful cases. We can circumvent this problem by analyzing the map ∂ of (2.2).

THEOREM 2.5. *Under our standing notation, with $I^2 \subseteq J \subset I$, the cokernel of $\Phi_J \to \Phi_I$ is isomorphic to a subquotient of $K_2(B/J)$. This subquotient is* $(\operatorname{Im} K_2(B/J, I/J) + \operatorname{Im} K_2(B))/(\operatorname{Im} K_2(A/J, I/J) + \operatorname{Im} K_2(B))$.

PROOF. A (third) diagram chase on (2.2) shows that $\operatorname{coker}(\Phi_J \to \Phi_I)$ is the subquotient $\operatorname{Im}(\partial)/\operatorname{Im}(\partial\eta)$ of $SK_1(B, J)$. From the diagram on p. 56 of **[Mil]**, we see that ∂ is the composite of the map $K_2(B/J, I/J) \to K_2(B/J)$ and the boundary map in the exact sequence containing $K_2(B) \to K_2(B/J) \to SK_1(B, J)$. Thus $\operatorname{Im}(\partial)/\operatorname{Im}(\partial\eta)$ is a subquotient of $K_2(B/J)/\operatorname{Im} K_2(B)$, and the result follows.

COROLLARY 2.6. *Φ_I is isomorphic to the following subquotient of $K_2(B/I^2)$:*

$$\frac{\operatorname{Im} K_2(B/I^2, I/I^2) + \operatorname{Im} K_2(B)}{\operatorname{Im} K_2(A/I^2, I/I^2) + \operatorname{Im} K_2(B)}.$$

3. Excision in the cusp. We now consider excision in a special case, providing counterexamples to the questions raised in the introduction.

The cusp case. We let k be a commutative ring of characteristic $p \neq 0$ and choose $N \geqslant n \geqslant 2$. We set $B = k[t]$, $A = k \oplus t^n B$, and $I = I_N = t^N B$.

The excision problem is thus parametrized by p, n, and N. Our philosophy is that if excision holds, one should be able to prove this using Mennicke symbols. Accordingly, we first indicate cases when excision holds.

Since $\Omega_{B/A} = B/(nt^{n-1}, t^n)dt$ **[GR2**, Lemma 2.3], the Swan-Vorst map shows that Φ_I is generated by $\varepsilon(\alpha t^l \otimes dt)$, $N \leqslant l < N + n$, $\alpha \in k$. ($l = N + n - 1$ is not needed unless $p|n$.)

LEMMA 3.1. *In the cusp case, $\varepsilon(\alpha t^l \otimes dt) = 0$ if $l \not\equiv -1 \pmod p$.*

PROOF.

$$(l+1)\varepsilon(\alpha t^l \otimes dt) = \begin{bmatrix} t^l \\ 1 - \alpha t^{l+1} \end{bmatrix}^{l+1} = \begin{bmatrix} t^{l+1} \\ 1 - \alpha t^{l+1} \end{bmatrix}^{l} = 1.$$

The result follows from the observation that $(l+1)\Phi_I = \Phi_I$.

COROLLARY 3.2. *If $n < p$ in the cusp case, excision holds for I_N when $N \equiv 0, 1, \ldots, p - n \pmod p$.*

PROOF. If $N \equiv 0, 1, \ldots, p - n \pmod p$, then $l \not\equiv -1 \pmod p$ for $N \leqslant l < N + n$ and $\Phi_I = 0$ by Lemma 3.1. If $N \equiv p - n \pmod p$, then $l \not\equiv -1 \pmod p$ for all $N \leqslant l \leqslant N + n - 2$. For $l = N + n - 1$ we have

$$n\varepsilon(\alpha t^{N+n-1} \otimes dt) = \begin{bmatrix} t^{N+n-1} \\ 1 - \alpha t^{N+n} \end{bmatrix}^{n} = \begin{bmatrix} t^{N} \\ 1 - \alpha t^{N+n} \end{bmatrix} \quad [\textbf{GR2}, \text{Lemma } 1.1] = 1$$

since $t^n \in A$. Since $n\Phi_I = \Phi_I$, the result follows. $\square$

We will now show that, for favorable k, excision fails under all other combinations of p, n, and N. For simplicity we now assume that k is a nonperfect field of characteristic p. The nonperfect assumption is necessary: when k is a perfect field, excision holds for all p, n, and N by [**GR2**, Theorem 4.3].

THEOREM 3.3. *In the cusp case with k a nonperfect field*:

(a) *If $n \geqslant p$, excision fails for all N.*

(b) *If $n < p$, excision holds if and only if $N \equiv 0, 1, \ldots, p - n \pmod p$.*

(c) *When excision fails, choose $\alpha \in k - k^p$ (so that $d\alpha \neq 0$ in Ω_k) and let $l \equiv -1 \pmod p$ be such that $N \leqslant l \leqslant N + n - 2$. If $n \equiv N \equiv 0 \pmod p$, we also allow $l = N + n - 1$. Then the symbol*

$$\varepsilon(\alpha t^l \otimes dt) = \begin{bmatrix} t^l \\ 1 - \alpha t^{l+1} \end{bmatrix}$$

is a nonzero element of $\Phi_{I_N} \subseteq SK_1(A, I_N)$.

REMARK. Part (b) gives the counterexample of §1 when $n = 2$.

REMARK. Theorem 3.3 and its proof hold for any commutative ring k satisfying $\Omega_k \neq 0$ and $K_i(k) = K_i(k[t])$, $i = 1, 2$, with α such that $d\alpha \neq 0$ in part (c).

REMARK. If $n \not\equiv 0 \pmod p$ and $N + n \equiv 0$, the computation in the proof of Corollary 3.2 shows that

$$\begin{bmatrix} t^{N+n-1} \\ 1 - \alpha t^{N+n} \end{bmatrix} = 1$$

for all α, even if $n > p$. This is a direct proof of the above observation that $l = N + n - 1$ is not needed unless $p|n$.

PROOF. The "if" part of (b) is Corollary 3.2. Note that in the remaining cases (i.e., either $n \geqslant p$ or both $n < p$ and $N \not\equiv 0, \ldots, p - n$) there is some $l \equiv -1 \pmod p$ in the range described by (c). Thus the theorem will follow if we can prove that $\varepsilon(\alpha t^l \otimes dt) \neq 0$ for l, α as described in part (c).

We will use the criterion of Theorem 2.5 with $I = I_N$ and $J = I_{l+1} = t^{l+1}B$. Note that $I^2 \subseteq J \subset I$. Recall from [**B**] that there is a map

$$\operatorname{dlog}\colon K_2(B/J) \to \Omega^2_{B/J} \cong (\Omega^2_k \otimes_k B/J) \oplus (\Omega_k \otimes_k B/J)dt.$$

Properties of the dlog map are given in [**B**, p. 206]. In particular, $\operatorname{dlog}(\langle f, g\rangle) = (1 + fg)^{-1} df \Lambda dg$. For manipulations of the second exterior power $\Omega^2_{B/J}$ of $\Omega_{B/J}$, the reader is referred to [**Gar**]. Note that as t^{2N} annihilates $\Omega^2_{B/J}$ we have

$$\operatorname{dlog}(\langle ft^N, gt^N\rangle) = (Nt^{2N-1})(df\Lambda gdt + fdt\Lambda dg).$$

This is zero since either $t^{2N-1} \in J$ or $2N = l + 1 = N + n$ in which case $n \equiv N \equiv 0 \pmod p$. As $\rho\langle ft^N, gt^N\rangle = (ft^N \otimes gt^N)$ (see the beginning of §2 for ρ) $\operatorname{dlog} \circ i\colon K_2(B/J, I/J) \to \Omega^2_{B/J}$ factors through the map ϕ of Lemma 2.1. We thus have a commutative diagram (whose rows are not exact):

$$\begin{array}{ccccccc}
K_2(A/J, I/J) & \to & K_2(B/J, I/J) & \xrightarrow{i} & K_2(B/J) & & \\
\downarrow\phi & & \downarrow\phi & & \downarrow\operatorname{dlog} & & \\
I/J \otimes_A \Omega_{A/I} & \to & I/J \otimes_B \Omega_{B/I} & \xrightarrow{\psi} & \Omega^2_{B/J} & \xrightarrow{\pi} & \Omega_k \otimes t^l dt
\end{array}$$

Here π is the k-module projection mapping $\Omega_k^2 \otimes B/J$ and $\Omega_k \otimes t^i dt$ to zero $(i \neq l)$.

Now $K_2(B) = K_2(k)$, so by the naturality of the dlog map [**B**, p. 206], the image of $K_2(B)$ in $K_2(B/J)$ maps into the subgroup Ω_k^2 of $\Omega_{B/J}^2$. Thus $\pi \circ$ dlog sends the image of $K_2(B)$ to zero.

We now show that the image of $K_2(A/J, I/J)$ is mapped to zero under $\pi \circ$ dlog. Indeed, $I/J \otimes_A \Omega_{A/I}$ is generated by symbols $\alpha t^i \otimes d\beta$ and $\alpha t^i \otimes dt^j$, where $\alpha, \beta \in k$, $N \leqslant i \leqslant l$ and $n \leqslant j \leqslant N - 1$. Since $2i \geqslant 2N \geqslant l + 1$ and $i + j \geqslant N + n \geqslant l + 1$,

$$\begin{aligned}\psi(\alpha t^i \otimes d\beta) &= \operatorname{dlog}\langle \alpha t^i, \beta\rangle = (1 + \alpha\beta t^i)^{-1}(t^i d\alpha \Lambda d\beta + i\alpha t^{i-1}\, dt \Lambda d\beta) \\ &= t^i d\alpha \Lambda d\beta + i\alpha t^{i-1}\, dt \Lambda d\beta - i\alpha^2\beta t^{2i-1}\, dt \Lambda d\beta, \\ \psi(\alpha t^i \otimes dt^j) &= \operatorname{dlog}\langle \alpha t^i, t^j\rangle = (1 + \alpha t^{i+j})^{-1}(j t^{i+j-1}\, d\alpha \Lambda dt) \\ &= t^{i+j-1}\, d\alpha \Lambda dt.\end{aligned}$$

If we apply π, we see that all these symbols map to zero, except when $l = N + n - 1$, in which case the following symbols are possibly nonzero:

$$\begin{aligned}\pi\psi(\alpha t^N \otimes d\beta) &= \alpha^2\beta N t^l d\beta \otimes dt \qquad (l = 2n - 1), \\ \pi\psi(\alpha t^N \otimes dt^n) &= n t^l d\alpha \otimes dt \qquad (l = N + n - 1).\end{aligned}$$

However, we allowed these cases only when $n \equiv N \equiv 0 \pmod p$, in which case these symbols also vanish.

Finally, since $\pi \circ$ dlog is zero on $\operatorname{Im} K_2(A/J, I/J) + \operatorname{Im} K_2(B)$, Proposition 2.3 and Theorem 2.5 tell us that $\varepsilon(\alpha t^l \otimes dt) \neq 0$ if $\pi \operatorname{dlog}(\langle \alpha t^l, t\rangle) \neq 0$. But $\operatorname{dlog}(\langle \alpha t^l, t\rangle) = t^l d\alpha \Lambda dt$. Thus $\pi \circ \operatorname{dlog}(\langle \alpha t^l, t\rangle) = d\alpha \otimes t^l dt$ is zero if and only if $d\alpha = 0$ in Ω_k. For a field of characteristic p, however, we have $d\alpha = 0$ if and only if $\alpha \in k^p$. This completes the proof.

REMARK. We can exactly identify the kernel Φ_I of excision when k is a (nonperfect) field of characteristic $p > 2$. It will be the direct sum of groups k/k^{p^r} over all $l \equiv -1 \pmod p$ in the range $N \leqslant l \leqslant N + n - 2$ (and including $l = N + n - 1$ if $n \equiv N \equiv 0$). Here r is such that $l + 1 = mp^r$ and $m \not\equiv 0 \pmod p$. The summand k/k^{p^r} corresponding to l is the subgroup of Mennicke symbols

$$\begin{bmatrix} t^l \\ 1 - \lambda t^{l+1} \end{bmatrix}, \qquad \lambda \in k - k^{p^r},$$

exactly as in part (c) of Theorem 3.3. The technical results needed to prove this are Theorem 4.1(iii) of [**B**, p. 236] and results on truncated polynomial rings in [**ST**]. Due to the length of the proof and its similarity to the proof of Theorem 3.3, it will not be given here.

Finally, we can use the solution of the excision problem for the ideals $t^N B$ to solve the excision problem for all ideals $ht^N B$ with $h \in A$. Since any ideal in A contains an ideal of the form $K = ht^N B$ for some $h(t) \in A$ such that $h(0) = 1$, part (b) of the following corollary gives counterexamples for question 2.

COROLLARY 3.4. *In the cusp case with k a nonperfect field, let $h(t)$ be a polynomial in A with constant term* 1.

(a) *If $n \geqslant p$, excision fails for all $K = ht^N B$.*

(b) *If $p > n$, excision holds for $K = ht^N B$ if and only if $N \equiv 0, 1, \ldots, p - n$* (mod p).

(c) *When excision fails, choose $\alpha \in k - k^p$ and let $l \equiv -1$* (mod p) *be such that $N \leqslant l \leqslant N + n - 2$ (include $l = N + n - 1$ if $n \equiv N \equiv 0$). Then the symbol*

$$\varepsilon(\alpha h t^l \otimes dt) = \begin{bmatrix} ht^l \\ 1 - \alpha h t^{l+1} \end{bmatrix}$$

is a nonzero element of Φ_k.

PROOF. We will show that, for $I = t^N B$, the natural map $\Phi_K \to \Phi_I$ is an isomorphism by showing that the obstruction groups given in Theorem 2.5 are naturally isomorphic. This will prove parts (a) and (b) of the corollary. To prove part (c) let $h = 1 + t^n b(t)$. Then in $I/I^2 \otimes \Omega_{B/A}$, $\alpha h t^l \otimes dt = \alpha t^l \otimes h dt = \alpha t^l \otimes (1 + t^n b) dt = \alpha t^l \otimes dt$ since $t^n dt = 0$. Therefore by part (c) of Theorem 3.3 the image of $\varepsilon(\alpha h t^l \otimes dt) \in SK_1(A, K)$ in $SK_1(A, I)$ is nonzero. This proves part (c).

Since (h^2) and $I^2 \cong t^{2N} B$ are relatively prime ideals in both A and B, $A/K^2 \simeq A/h^2 A \oplus A/I^2$ and $B/K^2 \simeq B/h^2 B \oplus B/I^2$. Note that this decomposition for B/K^2 can be done in such a way that $K_2(B)$ lies entirely in the second summand. We will compute the subquotient Φ_K of $K_2(B/K^2)$ relative to this decomposition. Since $K_2(B/K^2, K/K^2) \simeq K_2(B/h^2 B, hB/h^2 B) \oplus K_2(B/I^2, I/I^2)$ and $K_2(A/K^2, K/K^2)$ decomposes similarly, we have

$$\Phi_K \simeq \frac{\operatorname{Im} K_2(B/h^2 B, hB/h^2 B)}{\operatorname{Im} K_2(A/h^2 A, hA/h^2 A)} \oplus \frac{\operatorname{Im} K_2(B/I^2, I/I^2) + \operatorname{Im} K_2(B)}{\operatorname{Im} K_2(A/I^2, I/I^2) + \operatorname{Im} K_1(A)}.$$

The map $\Phi_K \to \Phi_I$ is projection onto the second summand. Thus we will be done if we can show that the map $A/h^2 A \to B/h^2 B$ is an isomorphism with $hA/h^2 A \simeq hB/h^2 B$. Since $h(0) = 1$ and $h \in A$, $h^2 B \cap A = h^2 A$. Thus $A/h^2 A$ maps injectively to $B/h^2 B$. In $B/h^2 B$ we have $t = (1 - t)(h^2 - 1) + 1$. But $(1 - t)(h^2 - 1) + 1 \in A/h^2 A$ since $h^2 - 1$ is in the conductor from B to A. Thus $A/h^2 A$ maps onto $B/h^2 B$ and $hA/h^2 A \simeq hB/h^2 B$. This completes the proof.

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