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ON FINITE GROUPS CONTAINING THREE CC-SUBGROUPS

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ABSTRACT. A finite group G has a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ if G contains three nonconjugate CC-subgroups A_1, A_2, A_3, such that $|N_G(A_1)| = 2|A_1|$, $|N_G(A_2)| = 4|A_2|$, $|N_G(A_3)| = 4|A_3|$. The authors prove that if a finite group G has a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ and $|G| < 3|A_1|^2|A_2|^2|A_3|^2$, then G has a nilpotent normal subgroup N such that G/N is isomorphic to $Sz(q)$ for suitable q.

Introduction. A proper subgroup A of a group G is called a *CC-subgroup* if $C_G(a) \leqslant A$ for all $a \in A^{\#}$. In this paper we shall consider a group G with a set A_1, A_2, A_3 of nonconjugate CC-subgroups. If $[N_G(A_i): A_i] = d_i > 1$, $1 \leqslant i \leqslant 3$, we shall describe this as a *self-centralization* system of type $(d_1|A_1|, d_2|A_2|, d_3|A_3|)$ or (d_1, d_2, d_3) for short.

The fact is that many simple groups have self-centralization systems. For example, J_4, J_{11} and M_{22} have types (22(23), 28(29), 10(31), 12(37), 14(43)), (6(7), 10(11), 6(19)) and (4(5), 3(7), 5(11)), respectively. Groups of Ree type have type (6, 6).

Several authors have studied groups with self-centralization systems of type (2). (See **[6]**, **[7]** and **[8]**.) It is an open question of Higman **[7]** whether or not there are finite simple groups of type (2, 4, 4) other than the Suzuki groups. Usami **[10]** answers this question under restrictions on the order of the CC-subgroups of G and the assumption that if r is an odd prime then an S_r-subgroup of G is abelian and $|G| \leqslant 3|A_1|^2|A_2|^2|A_3|^2$.

In this paper we shall prove the following theorem:

THEOREM A. *If G is a finite simple group with a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ and $|G| \leqslant 3|A_1|^2|A_2|^2|A_3|^2$, then G is isomorphic to one of the Suzuki groups.*

Theorem A has the following corollary.

COROLLARY A. *If G is a finite group with a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ and $|G| \leqslant 3|A_1|^2|A_2|^2|A_3|^2$, then G has a nilpotent normal subgroup N such that G/N is isomorphic to $Sz(q)$ for suitable q.*

Many papers appear in the literature which classify groups which have a character Λ of small degree relative to the orders of some subgroups of G. In a forthcoming paper, the authors will use Theorem A to classify all groups of type

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$(2n_1, 4n_2, 4n_3)$ which have a character Λ of relatively small degree compared to $n_1n_2n_3$.

1. Our notations are taken from **[5]** and character notation from **[6]**. If K is a nonempty set, let $|K|$ denote the number of elements in K. If p is a prime, then G_p denotes a Sylow p subgroup of G.

Let $G \in$ *Hypothesis* A if G is a finite simple group with a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$.

Let $|A_1| = n_1$, $|A_2| = n_2$ and $|A_3| = n_3$ where $n_2 < n_3$. Let $t_1 = (n_1 - 1)/2$, $t_2 = (n_2 - 1)/4$ and $t_3 = (n_3 - 1)/4$.

Let $G \in$ Hypothesis A, then Theorem 5.1 **[7]** implies G has one class of involutions. If τ is a fixed involution, then by conjugacy we may choose A_1, A_2, A_3 to be normalized by τ.

LEMMA 1. *Let $G \in$ Hypothesis* A, *then one of the following holds*:

(i) *G is isomorphic to $Sz(q)$.*

(ii)(a) $(n_1n_2n_3, 3) = 1$.

(b) *G contains elements of order* 2, 3, 4, *and* 6.

PROOF. Assume G is not isomorphic to a Suzuki group; then **[4]** and **[3]** imply G contains an element of order 6. Hence A_i a CC-subgroup for $i = 1, 2, 3$ implies $(n_1n_2n_3, 3) = 1$. Walter **[11]** implies G contains an element of order 4.

Let $G \in$ *Hypothesis* B if $G \in$ Hypothesis A, G is not isomorphic to a Suzuki group, and $|G| \leqslant 3|A_1|^2|A_2|^2|A_3|^2$.

Assume $G \in$ Hypothesis A, then $n_1 > 3$. Hence, there are t_1 irreducible characters X_i, $i = 1, \ldots, t_1$, of G which are associated with A_1 and one nonprincipal irreducible character Y such that $Y(z) \neq 0$ for $z \in A_1^{\#}$ **[6]**. Let 1_G denote the principal character of G. Lemma 3 **[6]** implies $X_1(1) = kn_1 + 2\varepsilon$ and $Y(1) = kn_1 + \varepsilon$ where k is a positive integer and $\varepsilon = \pm 1$. Lemma 6 **[6]** implies $|G| = n_1X_1(1)Y(1)(ln_1 + 1)$ where l is a nonnegative integer.

We incorporate some properties of X_1 and Y into Lemma 2.

LEMMA 2. *Assume $G \in$ Hypothesis* A *and let $T = \bigcup_{g \in G}(A_1^{\#})^g$, then the following hold*:

(i) *If $x \in G - T$, then $X_1(x)$ and $Y(x)$ are integers, and $X_1(x)Y(x)$ is a nonnegative integer.*

(ii) $\sum_{x \in G-T} X_1(x)Y(x) = |G|/n_1$.

(iii) $|C_G(\tau)|^2 = (ln_1 + 1)(Y(1) - Y(\tau))^2$.

(iv) $|Y(\tau)| \leqslant k + \varepsilon$.

(v) $(Y(1) - Y(\tau))^2 \geqslant k^2(n_1 - 1)^2$.

(vi) $(n_2n_3, X_1(1)) \neq 1$. *If $l = 0$, then $n_2n_3|X_1(1)$.*

PROOF. Since Y is the only nonprincipal nonexceptional irreducible character of G which does not vanish on $A_1^{\#}$, Y is equal to all its algebraic conjugates. Hence $Y(z)$ is an integer for all $z \in G$. Lemma 3 **[6]** implies if $x \in G - T$, then $X_1(x) = Y(x) + \varepsilon$. Hence $X_1(x)$ is an integer. Now $X_1(x)Y(x) = Y^2(x) + \varepsilon Y(x)$ and $Y(x)$ an integer implies (i).

The orthogonality relations imply $\Sigma_G X_1(g)Y(g) = 0$. Now $\Sigma_T X_1(g)Y(g) = (|G|/2n_1)(\Sigma_{g\in A_1^\#} \zeta_1(g))$ where ζ_1 is an irreducible character of $N_G(A_1)$ associated with X_1. $\Sigma_{A_1^\#} \zeta_1(g) = -2$; hence $\Sigma_{G-T} X_1(x)Y(x) = |G|/n_1$ and (ii) is proved.

Lemma 6 [**6**] implies (iii).

$N_G(A_1)$ a Frobenius group of order $2n_1$ implies there are t_1 irreducible characters ζ_i of $N_G(A)$ such that $\zeta_i(1) = 2$ and ζ_i vanishes off A_1. Let λ be a faithful linear character of $N_G(A_1)/A_1$, then $\{\lambda, \lambda^2, \zeta_i | i = 1, \ldots, t_1\}$ are all the irreducible characters of $N_G(A_1)$. Since Y is a nonexceptional character, it follows that $Y|_{N_G(A_1)} = r\lambda^2 + s\lambda + \Sigma_{i=1}^{t_1} k\zeta_i$ where $r + s = k + \varepsilon$. Now $\tau \in N(A_1)$ implies $Y(\tau) = r - s$; hence $|Y(\tau)| \leqslant k + \varepsilon$. Parts (iv) and (v) of the lemma follow.

Now $\Sigma_{A_1^\#} X_1(g)\overline{X_1(g)} = \varepsilon^2\Sigma_{A_1^\#} \zeta_1(g)\overline{\zeta_1(g)} = 2n_1 - 4$. Thus $(X_1, X_1) = |G|$ implies $\Sigma_{G-T} X_1(x)\overline{X_1(x)} + |G|(2n_1 - 4)/2n_1 = |G|$. Hence $\Sigma_{G-T} X_1(x)\overline{X_1(x)} = 2|G|/n_1$. Suppose $(X_1(1), n_2n_3) = 1$. X_1 is a nonexceptional character for $i = 2$ or 3. Hence, if $t_i \geqslant 2$ for $i = 2$ or 3, it follows that $|X_1(x)| = 1$ for all $x \in A_i^\#$. If $t_2 = 1$, then $n_2 = 5$ and all elements of order 5 are again conjugate. Hence (i) implies $|X_i(x)| = 1$ for all $x \in A_2^\#$. Thus $(n_2n_3, X_1(1)) = 1$ implies

$$|G|(n_2 - 1)/4n_2 + |G|(n_3 - 1)/4n_3 < |G|2/n_1.$$

Now $n_1 \geqslant 5$ implies $1/10 < 1/2 - 2/n_1 < 1/4n_2 + 1/4n_3 < 2/4n_2$. Hence $n_2 < 5$, which is a contradiction. Therefore, $(X_1(1), n_2n_3) \neq 1$.

If $l = 0$, then $|G_2|\,|X_1(1)$ or $|G_2|\,|Y(1)$. Theorem 17.4 [**4**] implies $X_1(\tau) = 0$ or $Y(\tau) = 0$. Lemma 3 [**6**] implies $Y(1) - Y(\tau) = X_1(1) - X_1(\tau)$, hence (iii) implies $|C_G(\tau)| = X_1(1)$ or $Y(1)$. Now $(n_2n_3, X_1(1)) \neq 1$ implies $|C_G(\tau)| = Y(1)$ so that $n_2n_3|X_1(1)$.

LEMMA 3. *Assume $G \in$ Hypothesis* B, *then $n_1 \geqslant 11$ and $n_2n_3|X_1(1)$.*

PROOF. Suppose $n_1 > 7$, then Lemma 1(ii)(a) implies $n_1 \geqslant 11$. Suppose $t_i \geqslant 2$ for $i = 2$ or 3, then $\{X_j | j = 1, \ldots, t_1\}$ does not contain any exceptional characters associated with $N(A_i)$. It follows that $X_j(z) = X_1(z)$ for $z \in A_i^\#$ and $t_1X_1(z)^2 \leqslant 3$. Now $n_1 \geqslant 11$ and Lemma 2(i) imply $X_1(z) = 0$. Hence $n_i|X_1(1)$. Suppose $n_2 = 5$, and let $z \in A_2^\#$. Since $X_j(z) = X_1(z)$ for $j = 1, \ldots, t_1$, the orthogonality relations imply $5X_1^2(z) \leqslant t_1X_1^2(z) \leqslant |C_G(z)| - 1 = 4$. Lemma 2(i) implies $X_1(z) = 0$. Thus $n_1 > 7$ implies $n_2n_3|X_1(1)$.

We will assume $n_1 \leqslant 7$ and obtain a contradiction. Let x_i, $i = 1, \ldots, t_1$, be representatives in $A_1^\#$ of the conjugacy classes of $A_1^\#$. Similarly let y_j, $j = 1, \ldots, t_2$, and $z_{\tilde{k}}$, $\tilde{k} = 1, \ldots, t_3$, be representatives of the conjugacy classes of $A_2^\#$ and $A_3^\#$. Let $B = \{x_i, y_j, z_{\tilde{k}} | i, j, \tilde{k}\}$, then $|B| = b = t_1 + t_2 + t_3$.

If $C_G(\tau)uC_G(\tau) \cap C_G(\tau)vC_G(\tau) \neq \varnothing$ for distinct elements $u, v \in B$, then there are elements w and $h \in C_G(\tau)$ such that $v = huw$. Conjugation by τ yields $v^{-1} = v^\tau = (huw)^\tau = hu^{-1}w$. It follows that $v = (v^\tau)^{-1} = (hu^{-1}w)^{-1} = w^{-1}uh^{-1}$. Hence, $v^2 = (w^{-1}uh^{-1})(huw) = w^{-1}u^2w$. Since all elements in B have odd order, $v^2 = u^{2w}$ implies that $|\langle u\rangle| = |\langle v\rangle| = k$. Now k odd implies $v = (v^2)(k + 1)/2 = (u^2)^{w((k+1)/2)} = (u^{2(k+1)/2})^w = u^w$. This contradicts the fact that u and v are not conjugate. Therefore, $C_G(\tau)uC_G(\tau) \cap C_G(\tau)vC_G(\tau) = \varnothing$ if u and v are distinct. If $w \in C_G(\tau) \cap C_G(\tau)^u$ where $u \in B$, then $w \in C_G(\tau) \cap C_G(\tau\tau^u) = C_G(\tau) \cap C_G(u^2)$

$= 1$. Therefore $|C_G(\tau)uC_G(\tau)| = |C_G(\tau)|^2$ if $u \in B$. It follows that $b|C_G(\tau)|^2 \leqslant |G|$. Lemma 6 [6] implies that $|G|(Y(1) - Y(\tau))^2 = n_1(kn_1 + \varepsilon)(kn_1 + 2\varepsilon)|C_G(\tau)|^2$. Hence, $b|C_G(\tau)|^2 \leqslant |G|$ implies that $b(Y(1) - Y(\tau))^2 \leqslant n_1(kn_1 + \varepsilon)(kn_1 + 2\varepsilon)$. Lemma 2(v) now implies

$$bk^2(n_1 - 1)^2 \leqslant n_1(kn_1 + 2\varepsilon)(kn_1 + \varepsilon). \tag{3.1}$$

Lemma 2(iii) implies that $n_2n_3|X_1(1)Y(1)$.

Suppose $n_1 = 5$, Lemma 1 implies $n_2 \geqslant 13$ and $n_3 \geqslant 17$, hence $b \geqslant 9$. Inequality (3.1) implies $19k^2 - 75k\varepsilon - 10 < 0$. Hence $\varepsilon = 1$ and $k \leqslant 4$. Direct computation of $X_1(1)Y(1)$ for $\varepsilon = 1$ and $1 \leqslant k \leqslant 4$ imply there are no n_2 and n_3 such that $n_2n_3|X_1(1)Y(1)$. Hence $n_1 > 5$.

Suppose $n_1 = 7$. If $k \leqslant 3$, then direct computation for $\varepsilon = \pm 1$ implies there are no n_2 and n_3 such that $n_2n_3|X_1(1)Y(1)$. Hence we may assume $k \geqslant 4$. Inequality (3.1) implies $b \leqslant 11$, hence $(n_2 - 1)/4 + (n_3 - 1)/4 \leqslant 11 - 3 = 8$. Calculation implies $(n_2, n_3) = (5, 13), (5, 17)$ or $(13, 17)$.

If $(n_2, n_3) = (13, 17)$, then $b = 10$ and (3.1) implies $17k^2 - 147k\varepsilon - 14 < 0$. Hence $\varepsilon = 1$ and $k \leqslant 9$. Direct computation implies $13 \cdot 17 \nmid X_1(1)Y(1)$ for $\varepsilon = 1$ and $k \leqslant 9$. Hence $(n_2, n_3) = (5, 13)$ or $(5, 17)$.

Suppose $l \neq 0$. Lemma 2(iii) implies $l7 + 1$ is a square, hence $l \geqslant 5$. Let $T \in \{X_1, Y\}$ where $n_3|T(1)$; then $T(1) = fn_3$. Since G satisfies Hypothesis B, $|G| \leqslant 3 \cdot 7^2 \cdot 5^2 \cdot n_3^2$. Now $3 \cdot 5^2 \cdot 7^2 \cdot n_3^2 \geqslant |G| \geqslant 7(fn_3 - 1)^2(5 \cdot 7 + 1)$ implies $f \leqslant 4$. If $n_3 = 17$ and $T = Y$, then $17f = \pm 1 \pmod 7$ implies $f = 2$ and $Y(1) = 34 = 5 \cdot 7 - 1$. Hence $X_1(1) = 33$ and $5 \nmid Y(1)X_1(1)$. If $n_3 = 17$ and $T = X_1$, then $17f = \pm 2 \pmod 7$ implies $f = 3$ and $X_1(1) = 7 \cdot 7 + 2$. Hence $Y(1) = 50$, but $n_2 = 5$ implies $50 \nmid |G|$. Therefore $n_3 \neq 17$. Similar calculations imply $n_3 \neq 13$.

Hence, we may assume $l = 0$. Lemma 2(vi) implies $X_1(1) = en_2n_3$. Now $7(en_2n_3 - 1)(en_2n_3) \leqslant |G| \leqslant 3 \cdot 7^2 \cdot 5^2 \cdot n_3^2$ implies $e \leqslant 4$. Since $4 \mid |G|$ and $|G_2| \mid Y(1)$, $e = 1$ or 3. However $17 \cdot 5 = 1 \pmod 7$ and $3 \cdot 17 \cdot 5 = 3 \pmod 7$ imply $n_3 \neq 17$. If $n_3 = 13$, then $5 \cdot 13 \cdot 3 = 6 \pmod 7$ implies $X_1(1) = 5 \cdot 13$ and $Y(1) = 64$. Hence $(|G|, 3) = 1$ and $G \notin$ Hypothesis B.

LEMMA 4. *Assume $G \in$ Hypothesis* B. *Let $U = \{Z \mid Z$ an irreducible character of G and $n_1n_2n_3 \nmid Z(1)\}$, then $|U| \leqslant 4 + t_1 + t_2 + t_3$.*

PROOF. If $n_2 > 5$, Lemma 4 [10] and Lemmas 1 and 3 imply the result. Hence we may assume $n_2 = 5$. Let x_i, $i = 2, 3$, denote elements of $A_2^\#$ and $A_3^\#$ respectively. Lemma 3 implies $n_2n_3|X_1(1)$. Lemma 3 [6] implies $Y(x_i) = -\varepsilon$ for $i = 2, 3$. Since all elements of order 5 are conjugate, the orthogonality relations imply there are 3 irreducible characters Z_1, Z_2 and Z_3 of G such that

$$|Z_i(x_2)| = 1. \tag{4.1}$$

If $V = \{1_G, Y, Z_1, Z_2, Z_3\}$, then $|V| = 5$ and V contains all the irreducible characters of G whose degrees are not divisible by 5.

Let $\{X_{3,j} \mid j = 1, \ldots, t_3\}$ be the exceptional characters of G associated with $N_G(A_3)$. The orthogonality relations imply there are 2 irreducible characters Z_5, Z_4

of G such that $|Z_5(x_3)| = |Z_4(x_3)| = 1$. If $W = \{1_G, Y, Z_4, Z_5, X_{3,j} | j = 1, \ldots, t_3\}$, then $|W| = t_3 + 4$ and W contains all the irreducible characters of G whose degrees are not divisible by n_3.

Now $Y(x_2)Y(x_3) + 1_G(x_2)1_G(x_3) = 2$ and the orthogonality relations imply

$$\sum_{i=1}^{3} Z_i(x_2)Z_i(x_3) = -2. \tag{4.2}$$

Hence $Z_i(x_3) \neq 0$ for some $i = 1$, 2, or 3. Now $X_{3,j}(x_2) = X_{3,1}(x_2)$ for $j = 1, \ldots, t_3$; hence $\{Z_i | i = 1, 2, 3\} \cap \{X_{3,j} | j = 1, \ldots, t_3\} \neq \varnothing$ implies $(n_3 - 1)/4 = 3$ and $\{Z_i | i = 1, 2, 3\} = \{X_{3,j} | j = 1, 2, 3\}$. Hence $U = \{1_G, Y, X_i, X_{3,j}, Z_4, Z_5 | i = 1, \ldots, t_1, j = 1, \ldots, t_3\}$ and $|U| = 3 + t_1 + t_2 + t_3$. If $\{Z_i | i = 1, 2, 3\} \cap \{X_{3,j} | j = 1, \ldots, t_3\} = \varnothing$, then (4.1) and (4.2) imply $\{Z_4, Z_5\} \subset \{Z_1, Z_2, Z_3\}$. Hence $U = \{1_G, Y, Z_1, Z_2, Z_3, X_i, X_{3,j} | i = 1, \ldots, t_1, j = 1, \ldots, t_3\}$ so that $|U| = t_1 + t_2 + t_3 + 4$.

LEMMA 5. *Assume $G \in$ Hypothesis* B, *then $|G| > (n_1n_2n_3)^2$. If r is an odd prime which is relatively prime to $3n_1n_2n_3$ and $r | |G|$, then $r = 5$ and G_5 is an elementary abelian CC-subgroup of order* 25.

PROOF. Let t be the number of conjugacy classes of elements in G which do not meet $(A_1 \cup A_2 \cup A_3)^{\#}$. Lemma 1(ii) implies $t = 5 + j$ where $j > 0$. Lemma 4 implies there are at least $j + 1$ irreducible characters Z of G such that $n_1n_2n_3 | Z(1)$. Hence $(j + 1)(n_1n_2n_3)^2 < |G|$ implies $|G| > (n_1n_2n_3)^2$ and $j \leqslant 1$.

Let r be as in the hypothesis, then $j = 1$. Hence, centralizers of elements of order r are G_r groups. Therefore, G_r is a CC-subgroup of G [9]. Since all elements of $G_r^{\#}$ are conjugate, G_r is elementary abelian, and $C_G(G_r) = G_r$. Now $j = 1$ and Lemma 1 imply G contains no elements of order 12 or 8.

$N_G(G_r)$ is a Frobenius group with G_r as a Frobenius kernel. Let L be the Frobenius complement. Clearly $|L| = |G_r^{\#}|$. Since L is a Frobenius complement, an S_2-subgroup of L is cyclic or generalized quaternion and the Sylow subgroups of L of odd prime order are cyclic. In this case, the highest power of 2 dividing $|L|$ is 2 or 4 or 8 by Lemma 1. If an odd prime r' divides $|L|$, it is well known that $2 | |Z(L)|$. Hence $r' = 3$. Hence $|L| = 2, 4, 8, 6, 12$, or 24. Consequently, $|G_r| = 3, 5, 9, 7, 13$, or 5^2. The hypothesis implies $(|G_r|, 3) = 1$; hence $|G_r| = 5, 7, 13, 5^2$.

On the other hand, let y be an element of order r. Then $N_G(\langle y \rangle)/C_G(y)$ is a cyclic group of order $r - 1$ since the elements of order r form a single class. Therefore, the possibilities of the order of a cyclic subgroup of even order in G are 2, 4, 6 and $|G_r| = 5$, 7 or 5^2. Let x be an element of order r. The class equation implies

$$|G| \geqslant 1 + \frac{|G|}{|G_r|} + \frac{|G|(n_1 - 1)}{2n_1} + \frac{|G|(n_2 - 1)}{4n_2} + \frac{|G|(n_3 - 1)}{4n_3}.$$

Hence $1/2n_1 + 1/4n_2 + 1/4n_3 \geqslant 1/|G_r|$. Lemma 3 implies $n_1 \geqslant 11$. Now $n_2 \geqslant 5$ and $n_3 \geqslant 13$ imply $|G_r| > 7$.

LEMMA 6. *Assume $G \in$ Hypothesis* B, *then the following hold*:
(i) $X_1(1) = n_2n_3$.
(ii) $ln_1 + 1 = n_1 + 1$ *or* $3n_1 + 1$.
(iii) $ln_1 + 1 = 2^{2a}3^{2b}$ *where a is a positive integer and b is a nonnegative integer.*

PROOF. Lemma 3 implies $X_1(1) = en_2n_3$, where e is a positive integer.

Suppose $l = 0$. As in the proof of Lemma 2, we see that $|C_G(\tau)| = Y(1)$. Lemma 5 implies $(n_1n_2n_3)^2 < n_1(en_2n_3)(en_2n_3 + 1)$, hence $e > 1$. Lemma 1 implies $3 \mid |C_G(\tau)|$, hence $(X_1(1), Y(1)) = 1$ implies $|G_3| \mid Y(1)$. Therefore $X_1(1) = en_2n_3$ where e is odd, $(e, 3n_1n_2n_3) = 1$, and $e > 1$. Lemma 5 implies $(|C_G(\tau)|, 5) = 5$ and $X_1(1) = 25n_2n_3$. Hence $X_1(1) = 1 \pmod 4$. Since $4 \mid Y(1)$, $Y(1) = X_1(1) - \varepsilon$ implies $\varepsilon = 1$. This contradicts Theorem A [2]. Hence $l \geqslant 1$.

Since $l \neq 0$,

$$n_1(en_2n_3)(en_2n_3 - 1)(ln_1 + 1) \leqslant |G| \leqslant 3(n_1n_2n_3)^2$$

implies $le^2 < 4$. Hence $e = 1$ and $l \leqslant 3$. Lemma 2(iii) implies $ln_1 + 1$ is a square, hence $l = 1$ or 3. Lemmas 1, 5 and 2(iii) imply $ln_1 + 1 = 2^{2a}3^{2b}$, where $a \geqslant 1$.

LEMMA 7. *Assume $G \in$ Hypothesis* B, *then $ln_1 + 1 = 2^{2a}3^{2b}$ where a and b are positive integers.*

PROOF. Lemma 6(iii) implies a is a positive integer. Suppose $b = 0$. Lemmas 1 and 5 imply $|C_G(\tau)| = |G_2|3^c$ where $c \geqslant 1$. Lemma 6 implies $X_1(1) = n_2n_3$. Hence $ln_1 + 1 = 2^{2a}$ implies $|G_3| \mid Y(1)$. Lemma 2(iii) implies $(Y(1)_2)^2(ln_1 + 1)^2 3^{2c} = |C_G(\tau)|^2 = (ln_1 + 1)(Y(1) - Y(\tau))^2$. Valuation Theory implies $|Y(\tau)|_2 = Y(1)_2$ and $|Y(\tau)|_3 = 3^c$. Lemma 2(iv) now implies

$$|C_G(\tau)| = |Y(\tau)|_2 |Y(\tau)|_3 (ln_1 + 1) \leqslant |Y(\tau)|(ln_1 + 1) \leqslant (k + 1)(ln_1 + 1).$$

Lemma 2(iii) and (v) imply $|C_G(\tau)|^2 \geqslant k^2(n_1 - 1)^2(ln_1 + 1)$. It follows that

$$(k + 1)^2(ln + 1)^2 \geqslant |C_G(\tau)|^2 \geqslant k^2(n_1 - 1)^2(ln + 1).$$

Now $l \leqslant 3$ and $k \geqslant 1$ imply $4(3n_1 + 1) \geqslant (n_1 - 1)^2$. Hence $n_1 < 15$. Lemmas 1 and 3 imply $n_1 = 11$ or 13. However $n_1 + 1 \neq 2^{2a}$ if $l = 1$ or 3 and $n_1 = 11$ or 13.

PROOF OF THEOREM A. We will assume $G \in$ Hypothesis B and obtain a contradiction. Lemma 7 implies $|G_2| \nmid Y(1)$, hence Theorem 17.4 [1] implies there is an element y such that $|\langle y \rangle| = 2^k > 1$ and $Y(y) \neq 0$. Since Y is integer valued, $Y(y^i) = Y(y)$ if $(i, 2) = 1$. Lemma 6 implies $X_1(1) = n_2n_3$. Lemma 2 implies $X_1(z)$ is an integer for $z \in G - \bigcup_g (A_1)^g$. Hence $(n_2n_3, 2) = 1$ implies $X_1(y^i) \neq 0$ for any i.

Let $R = \{y^g \mid g \in G\}$, $n = |G|/|G_2|$ and $U_R = \{x \mid x^n \in R\}$. Clearly $\bigcup_R$ is a union of conjugacy classes. Since $(n, 8) = 1$, there is an odd integer i_0 such that $y^{i_0 n} = y$. Let $z_1 = y^{i_0}$, Lemma 2(i) implies $X_1(z_1)Y(z_1)$ is a positive integer. Let $z_1, z_2, \ldots, z_t$ be representatives of the conjugacy class in $\bigcup_R$. Now X_1Y is a character of G, hence Theorem 17.3 [1] implies

(a) $(|G|, n)^{-1}\Sigma_{\bigcup_R} X_1(x)Y(x) = c$ where c is an algebraic integer. Since $\bigcup_R \leqslant G - T$, Lemma 2(i) and $X_1(z_1)Y(z_1) \neq 0$ imply c is a positive rational number. Hence c is a positive integer.

$|G_2| = Y(1)_2(ln_1 + 1)_2$ and (a) imply (b) $\Sigma_{\cup_R} X_1(x)Y(x) = c(|G|, n) = c|G|/|G_2| = c|G|/Y(1)_2(ln_1 + 1)_2$. Now

$$\sum_{\cup_R} X_1(x)Y(x) = \sum_{i=1}^{t} \frac{X_1(z_i)Y(z_i)|G|}{|C_G(z_i)|}$$

and the orthogonality relations imply

$$Y(1)\left|\frac{X_1(z_i)Y(z_i)|G|}{|C_G(z_i)|}\right. \quad \text{for } i = 1, \ldots, t.$$

Equation (b) now implies

$$Y(1)\left|\frac{c|G|}{Y(1)_2(ln_1 + 1)_2}\right. = \frac{c|G|}{|G|_2}.$$

Hence $c = Y(1)_2 d$ where d is a positive integer. Thus

$$\sum_{\cup_R} X_1(x)Y(x) = dG/(ln_1 + 1)_2.$$

Lemma 2(ii) now implies $dG/(ln_1 + 1)_2 < |G|/n_1$ so that $(ln_1 + 1)_2 > n_1$. Lemma 7 implies $ln_1 + 1 \geqslant 9(ln_1 + 1)_2 > 9n_1$. Hence $l > 3$ which contradicts Lemma 6.

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