A CLASSIFICATION THEOREM FOR SKT-MODULES

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Abstract. In this paper a class of Abelian groups which includes the torsion totally projective groups, S-groups, and balanced projectives is studied. It is shown that this class of groups has a complete set of invariants.

If $p$ is a prime number then the ring of all rational numbers $a/b$ with $b$ relatively prime to $p$ will be denoted by $\mathbb{Z}_p$.

The category of $\mathbb{Z}_p$-modules are those Abelian groups with the property that multiplication by a prime other than the prime $p$ is an automorphism of the group. If $M$ is a $\mathbb{Z}_p$-module then (i) the torsion submodule of $M$ is the maximal torsion subgroup of $M$; (ii) $M$ is torsion-free if and only if the torsion submodule of $M$ is $(0)$; (iii) $M$ is reduced if and only if $M$ is a reduced group; and (iv) $M$ is divisible if and only if $M$ is a divisible group.

$\mathbb{Z}$ and $\mathbb{Q}$ will denote the groups of integers and rational numbers respectively. The group $\text{Ext}(\mathbb{Q}/\mathbb{Z}_p, \ast)$ will be denoted by $c(\ast)$.

The limit ordinal $\lambda$ is a limited ordinal cofinal with $\omega$ if there is a sequence of smaller ordinals $\beta_i$ such that $\lambda = \sup \beta_i$. Otherwise $\lambda$ is said to be not cofinal with $\omega$.

Three families of invariants $u$, $h$, and $k$ are defined for the group $G$ as follows. The invariant $u(p^\alpha, G)$ (the Ulm invariant) for the prime $p$ and ordinal $\alpha$ is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-vector space $(p^\alpha G)[p]/(p^{\alpha+1}G)[p] \equiv U(p^\alpha, G)$. The invariant $h(p^\beta, G)$ for the prime $p$ and ordinal $\beta$ is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-vector space $p^\beta G/(p^\beta+T)$, where $T$ is the maximal torsion subgroup of $p^\beta G$.

The invariant $k(p^\lambda, G)$ for the prime $p$ and limit ordinal $\lambda$ such that $\lambda$ is not cofinal with $\omega$, is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-vector space $K(p^\lambda, G) \equiv p^\lambda c(G/p^\lambda G)/p^{\lambda+1}c(G/p^\lambda G)$.

Warfield [2] and [3] showed that the family of invariants $u$ and $h$ classify the balanced projectives, and the family of invariants $u$ and $k$ classify the $S$-groups. Noting that the family of invariants $h$ are 0 on the $S$-groups, and the family of invariants $k$ are 0 on the balanced projectives, it was conjectured that the family of invariants $u$, $h$ and $k$ could be used to classify those groups which are the direct sum of $S$-groups and balanced projectives. These groups are called SKT-modules. It will be shown in [4] that SKT-modules are projective relative to a well-defined class of sequences. It then follows that the SKT-modules form a class of groups.
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which contains the S-groups and balanced projectives, have a projective characterization, and a complete family of invariants.

For the limit ordinal \( \lambda \), a \( \mathbb{Z}_p \)-module \( M \) is a \( \lambda \)-elementary balanced projective if and only if \( p^\lambda M \cong \mathbb{Z}_p \) and \( M/p^\lambda M \) is a totally projective \( p \)-group. The torsion subgroup of \( M \) is called a \( \lambda \)-elementary \( S \)-group. A \( \mathbb{Z}_p \)-module \( M \) is a balanced projective if and only if it is isomorphic to the direct sum of a totally projective \( p \)-group and \( \lambda \)-elementary balanced projectives for various limit ordinals \( \lambda \). The torsion subgroup of a balanced projective is called an \( S \)-group.

**Definition.** A module \( M \) is an SKT-module if and only if there is a balanced projective \( K \) and an \( S \)-group \( S \) such that \( M \) is isomorphic to \( K \oplus S \).

**Lemma.** If \( K \) is a balanced projective and \( \lambda \) is a limit ordinal not cofinal with \( \omega \), then \( k(p^\lambda, K) = 0 \).

**Proof.** Write

\[ K \cong T \oplus \left( \bigoplus_{\beta \in \Gamma} M_\beta \right) \]

where \( T \) is a totally projective \( p \)-group and \( M_\beta \) is the direct sum of \( \beta \)-elementary balanced projectives. Let \( \lambda \) be a limit ordinal such that \( \lambda \) is not cofinal with \( \omega \). If \( K \cong A_\lambda \oplus B_\lambda \) where \( A_\lambda \) is the direct sum of \( T \) and \( M_\beta \) where \( \beta > \lambda \), and \( B_\lambda \) is the direct sum of all \( M_\beta \) where \( \beta < \lambda \), then \( K/p^\lambda K \cong A_\lambda/p^\lambda A_\lambda \oplus B_\lambda \). Since \( A_\lambda/p^\lambda A_\lambda \) is a totally projective \( p \)-group, \( p^\lambda c(A_\lambda/p^\lambda A_\lambda) = 0 \) by [1, 3.10]. If \( S \) is the maximal torsion subgroup of \( B_\lambda \) then \( S \) is the direct sum of \( p \)-groups of length less than \( \lambda \) and \( 0 = p^\lambda c(S) = p^\lambda c(B_\lambda) \) by [1, 3.10]. It has therefore been shown that \( p^\lambda c(K/p^\lambda K) = 0 \) and \( k(p^\lambda, K) = 0 \).

**Theorem.** If \( A \) and \( B \) are SKT-modules such that \( u(p^\alpha, A) = u(p^\alpha, B) \), \( h(p^\beta, A) = h(p^\beta, B) \), and \( k(p^\lambda, A) = k(p^\lambda, B) \) for all ordinals \( \alpha, \beta \) and \( \lambda \) such that \( \beta \) and \( \lambda \) are limit ordinals and \( \lambda \) is not cofinal with \( \omega \), then \( A \) is isomorphic to \( B \).

**Proof.** Since \( A \) and \( B \) are SKT-modules, \( A \cong K \oplus S \) and \( B \cong K \oplus S \) where \( K \) and \( K \) are balanced projectives, and \( S \) and \( S \) are \( S \)-groups. It can be assumed that \( S \cong \bigoplus_{\lambda \in \Lambda} S_\lambda \) and \( S \cong \bigoplus_{\lambda \in \Delta} S_\lambda \) where \( S_\lambda \) and \( S_\lambda \) are the direct sum of \( \lambda \)-elementary \( S \)-groups, and if \( \lambda \in \Lambda \) or \( \lambda \in \Delta \) then \( \lambda \) is not cofinal with \( \omega \). Since \( k(p^\lambda, A) = k(p^\lambda, S) = k(p^\lambda, B) = k(p^\lambda, S) \), \( \Lambda = \Delta = \nabla \). For each \( \lambda \in \nabla \), let \( S_\lambda \) and \( S_\lambda \) be the torsion subgroups of the balanced projectives \( N_\lambda \) and \( N_\lambda \) respectively. Furthermore, it will be assumed that \( N_\lambda \) and \( N_\lambda \) are direct sums of \( \lambda \)-elementary balanced projectives. Note that for each \( \lambda \in \nabla \), \( p^\lambda N_\lambda \cong p^\lambda N_\lambda \) because \( h(p^\lambda, N_\lambda) = k(p^\lambda, S_\lambda) = k(p^\lambda, A) = k(p^\lambda, B) = k(p^\lambda, S_\lambda) = h(p^\lambda, N_\lambda) \), [3, 2.3]. Let \( M \cong K \oplus (\bigoplus_{\lambda \in \nabla} N_\lambda) \) and \( M \cong K \oplus (\bigoplus_{\lambda \in \nabla} N_\lambda) \). Since \( K \) and \( K \) are balanced projectives, then \( K \cong T \oplus (\bigoplus_{\beta \in \Omega} K_\beta) \) and \( K \cong T \oplus (\bigoplus_{\beta \in \psi} K_\beta) \), where \( T \) and \( T \) are totally projective \( p \)-groups, and \( K_\beta \) and \( K_\beta \) are the direct sum of \( \beta \)-elementary balanced projectives. Since \( h(p^\beta, K) = h(p^\beta, A) = h(p^\beta, K) = h(p^\beta, K) \), \( \nabla \cong \psi \cong \Omega \), furthermore, if \( \beta \in \Omega \) then \( p^\beta K_\beta \cong p^\beta K_\beta \).
Let

\[ N = \left( \bigoplus_{\beta \in \Omega} p^\beta K_{\beta} \right) \oplus \left( \bigoplus_{\lambda \in \Lambda} p^\lambda N_{\lambda} \right) \quad \text{and} \quad \hat{N} = \left( \bigoplus_{\beta \in \Omega} p^\beta K_{\beta} \right) \oplus \left( \bigoplus_{\lambda \in \Lambda} p^\lambda N_{\lambda} \right). \]

Then there is a height preserving isomorphism \( \phi: N \to \hat{N} \) such that \( \phi \) takes \( p^\beta K_{\beta} \) onto \( p^\beta K_{\beta} \) and \( p^\lambda N_{\lambda} \) onto \( p^\lambda N_{\lambda} \) for each \( \beta \in \Omega \) and \( \lambda \in \Lambda \). Note that \( N \) is a \( p \)-nice submodule of \( M \) and \( \hat{N} \) is a \( p \)-nice submodule of \( M \). In addition, \( M/N \) and \( M/\hat{N} \) are totally projective \( p \)-groups. For each \( \alpha \), \( U(p^\alpha, M)/I(p^\alpha, N) \cong U(p^\alpha, M)/I(p^\alpha, N) \) since \( M \) and \( M \) have the same Ulm invariants and \( I(p^\alpha, N) = I(p^\alpha, N) = 0 \). The map \( \phi \) extends to an isomorphism \( \phi^* \) of \( M \) onto \( \hat{M} \) [3, 1.2]. The module \( A \) can be identified with the submodule consisting of all elements \( g \) in \( M \) such that for some integer \( n \), \( p^ng \) is in the submodule \( \bigoplus_{\beta \in \Omega} p^\beta K_{\beta} \). The module \( B \) can be identified with a similar submodule of \( M \). Since the isomorphism \( \phi^* \) takes the submodule \( \bigoplus_{\beta \in \Omega} p^\beta K_{\beta} \) onto \( \bigoplus_{\beta \in \Omega} p^\beta K_{\beta} \), \( \phi^* \) will take \( A \) onto \( B \).

**References**


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