TAME MEASURES ON CERTAIN COMPACT SETS

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ABSTRACT. A finite complex Borel measure $\mu$ on a compact subset $X \subset \mathbb{C}^n$ is called tame if there exist finite measures $\sigma_1, \ldots, \sigma_n$ on $X$ with

$$\int_X \phi \, d\mu = \int_X \sum_{j=1}^n \frac{\partial \phi}{\partial z_j} \, d\sigma_j$$

for every $\phi \in C_0^\infty(\mathbb{C}^n)$. We define $X_T = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1$ and $z_1 \in T\}$, where $T$ is a compact subset of $\{|z| < 1\}$ in $\mathbb{C}$. It is shown in this paper that tame measures form a weak-* dense subset of $\mathcal{M}(X_T)$. It follows then, with the help of a theorem by Weinstock, that $R(X_T)$ is a local algebra.

Let $X$ be a compact set in $\mathbb{C}^n$. $C(X)$ is the algebra of all continuous functions on $X$. $R_0(X)$ is the algebra of all rational functions $P/Q$ on $\mathbb{C}^n$ with $P, Q$ polynomials and $Q \neq 0$ on $X$. $R(X)$ is the uniform closure of $R_0(X)$ in $C(X)$.

It is a well-known consequence of Cauchy-Green formula that if $\mu$ is a complex Borel measure with compact support $X \subset \mathbb{C}$, then

$$\int \phi \, d\mu = -\frac{1}{2\pi i} \int \frac{\partial \phi}{\partial \bar{z}} \left( \int \frac{1}{\xi - z} \, d\mu(\xi) \right) \, dz \wedge d\bar{z}$$

holds for every $\phi \in C_0^\infty(\mathbb{C})$. It follows that $\mu$ is an orthogonal measure for $R(X)$ iff

$$\hat{\mu} = \int \frac{1}{\xi - z} \, d\mu(\xi)$$

is supported on $X$, or, equivalently, the measure $\hat{\mu}(z)dz \wedge d\bar{z}$ is supported on $X$. This gives a description of orthogonal measures for $R(X)$ where $X \subset \mathbb{C}$. While no general description for measures on $X \subset \mathbb{C}^n$, $n > 1$, orthogonal to $R(X)$ is available, we introduce the following definition.

DEFINITION. Let $X$ be a compact set in $\mathbb{C}^n$. A finite complex Borel measure is tame if there exist finite measures $\sigma_1, \ldots, \sigma_n$ on $X$ with

$$\int_X \phi \, d\mu = \int_X \sum_{j=1}^n \frac{\partial \phi}{\partial z_j} \, d\sigma_j$$

for every $\phi \in C_0^\infty(\mathbb{C}^n)$. (1)

(1) and (2) now imply that for $X \subset \mathbb{C}$ a measure $\mu$ on $X$ is orthogonal to $R(X)$ iff $\mu$ is tame.

Let $X \subset \mathbb{C}^n$ be fixed. Suppose that a tame measure $\mu$ exists on $X$ with $\mu \neq 0$. If $\phi \in R_0(X)$ then $\partial \phi/\partial z_j \equiv 0$ on $X$ for all $j$. So by (1) $\int_X \phi \, d\mu = 0$. It follows that $\mu \perp R(X)$ and hence $R(X) \neq C(X)$. Thus the existence of tame measures imply that $R(X) \neq C(X)$.

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In this paper, we restrict ourselves to subsets of $\partial B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1\}$ which have the form $X_T = \{(z_1, z_2) \in \partial B: z_1 \in T\}$ where $T$ is a compact subset of $\{|z_1| < 1\}$ in $\mathbb{C}$. We study orthogonal measures for $R(X_T)$ and problems related to these measures.

Basener [1] has constructed a compact subset $\tilde{X}_T$ (which has the above stated form) of $\partial B$ such that $\tilde{X}_T$ is rationally convex, yet $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. In the following, we will construct an ample family of tame measures for $R(X_T)$, provided that $R(T) \neq C(T)$. In fact, they form a weak-* dense set of $R(X_T)^\perp$. This gives an alternative explanation why $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. Moreover, the weak-* density along with a theorem of Weinstock [2] lead to the conclusion that $R(X_T)$ is a local algebra. I.e., if $\{U_\alpha\}$ is a finite open covering of $X_T$ and if $f \in C(X_T)$ is such that $f|_{X_T \cap U_\alpha}$ is in $R(X_T \cap U_\alpha)$, for all $\alpha$, then $f \in R(X_T)$. The main results may be stated as follows.

**Theorem 1.** Assume that $R(T) \neq C(T)$. Then the set of tame measures on $X_T$ is weak-* dense in the set of all orthogonal measures to $R(X_T)$ on $X_T$.

**Theorem 2.** Let $\phi$ be a smooth function with $\partial \phi / \partial z_i \equiv 0$ on $X_T$, $i = 1, 2$. Then $\phi \in R(X_T)$.

**Theorem 3.** $R(X_T)$ is a local algebra.

**Notations.**

$$
B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 < 1\}, \\
\partial B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1\}, \\
\Delta = \{z_1 \in \mathbb{C}: |z_1| < 1\}, \\
X_T = \{(z_1, z_2) \in \partial B, z_1 \in T\} \text{ where } T \text{ is a compact subset of } \Delta, \\
\Gamma_{z_1} = \left\{\left(z_1, (1 - z_1 \bar{z}_1)^{1/2} e^{i\theta}\right): -\pi < \theta < \pi\right\}.
$$

Let $\phi$ be any smooth function in a neighborhood of $X_T$. Let $\tilde{\phi}$ denote the composite $\phi \circ p$ where $p$ is the map from $\{|z_1| < 1\} \times [-\pi, \pi]$ to $\partial B$ defined by $p(z_1, \theta) = (z_1, (1 - z_1 \bar{z}_1)^{1/2} e^{i\theta})$. For each fixed $z_1 \in T$, $\tilde{\phi}$ has the following Fourier expansion on $\Gamma_{z_1}$:

$$
\tilde{\phi}(z_1, z_2) = \phi(z_1, \theta) = \sum_{-\infty}^{\infty} \phi_n(z_1) e^{imb}, \quad z_2 = (1 - z_1 \bar{z}_1)^{1/2} e^{i\theta},
$$

where

$$
\phi_n(z_1) = \int_{-\pi}^{\pi} \phi(z_1, \theta) e^{-imb} \frac{dt}{2\pi}
$$

is the $n$th Fourier coefficient of $\phi(z_1, \theta)$.

It is well known that

(i) $\phi_n(z_1)$ is smooth in $z_1$,

(ii) if $n \neq 0$, $|\phi_n(z_1)| \leq M/n^3$ for all $z_1 \in T$ where $M$ is a constant depending on $\phi$. 


**Theorem 1.** Assume that $R(T) \neq C(T)$. Then the set of tame measures on $X$ is weak-* dense in the set $R(X_T)^\perp$ of all measures on $X_T$ orthogonal to $R(X_T)$.

**Proof.** Let $\nu$ be a nonzero orthogonal measure for $R(T)$. Consider the linear functional which assigns to each $f$ in $C(X_T)$ the value

$$ \int_T \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1). $$

Since $T$ is a compact subset of $\Delta$, we have $X_T \cap \{z_2 = 0\} = \emptyset$. Hence the above is well defined. It is easy to see that this linear functional is continuous, therefore it defines a measure $\nu$ on $X_T$, i.e.

$$ \int f \, d\nu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1) \quad \text{for all } f \in C(X_T). \quad (***) $$

**Assertion.** $\nu$ is tame.

Let $\phi \in C_0^{\infty}(C^2)$,

$$ \int_{X_T} \phi \, d\nu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} \phi(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1) = \int_T \phi_0(z_1) \, d\nu(z_1) = \frac{1}{2\pi} \int -\frac{\partial \phi_0}{\partial \xi_1} \tilde{\nu}(\xi_1) \, d\xi_1 \wedge d\xi_1 \text{ by (1).} $$

Assume the following lemma which will be proved later.

**Lemma.**

$$ \frac{\partial \phi_0}{\partial \xi_1} = \left( \frac{\partial \phi}{\partial \xi_1} - \frac{\xi_1}{\xi_2} \frac{\partial \phi}{\partial \xi_2} \right) \bigg|_{0} \tag{3} $$

the zeroth Fourier coefficient of $\partial \phi/\partial \xi_1 - (\xi_1/\xi_2)(\partial \phi/\partial \xi_2)$.

We get that

$$ \int_{X_T} \phi \, d\nu = -\frac{1}{2\pi i} \int \left( \frac{\partial \phi}{\partial \xi_1} - \frac{\xi_1}{\xi_2} \frac{\partial \phi}{\partial \xi_2} \right) \tilde{\nu}(\xi_1) \, d\xi_1 \wedge d\xi_1 $$

Let $\sigma_1$ be the measure on $\partial B$ such that for $f$ in $C(\partial B)$,

$$ \int f(z_1, z_2) \, d\sigma_1 = \int \frac{1}{4\pi^2} \left( \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) \tilde{\nu}(\xi_1) \, dz_1 \wedge d\xi_1 $$

and let $\sigma_2 = -(z_1/z_2)\sigma_1$. 


Again, the above definitions are legitimate, for \( \nu \perp R(T) \) implies that \( \tilde{\nu}(z) = 0 \) outside \( T \). This also shows that \( \sigma_1, \sigma_2 \) are supported on \( X_T \). To sum up, we have shown that for any \( \phi \in C_0^\infty(\mathbb{C}^n) \),

\[
\int \phi \, d\mu = \int \frac{\partial \phi}{\partial \bar{z}_1} \, da_1 + \int \frac{\partial \phi}{\partial \bar{z}_2} \, da_2
\]

where \( \sigma_i \)'s are supported on \( X_T \). Hence \( \mu \) is tame. \( \mu \) is not a zero measure because \( \int f(z) \, d\mu = \int f(z) \, d\nu \) for all \( f \in C(\mathbb{C}^n) \) and \( \nu \) is nonzero by hypothesis.

We note that if \( \mu \) is a tame measure on \( X \subset \mathbb{C}^n \), then \( f \mu \) is also tame for smooth function \( f \) with \( \partial f/\partial \bar{z}_i \equiv 0 \) on \( X \), \( i = 1, \ldots, n \). In particular, if \( \mu \) is as in (**), the measures \( \tilde{z}_j^m \mu \), \( m = \pm 1, \pm 2, \ldots \), are all tame. Let \( S = \{ z^m \mu : \text{there is a nonzero orthogonal measure } \nu \text{ for } R(T) \text{ such that } \nu \text{ is defined by (**), } m = 0, \pm 1, \pm 2, \ldots \} \).

We will show that

"If \( f \) in \( C(X_T) \) is such that \( f \) is annihilated by all elements in \( S \), then \( f \) is in \( R(X_T) \)."

Let \( \sum_{\infty} f_n(z_1) e^{inz_1} \) be the "formal" Fourier expansion for \( \tilde{f}(z, \theta) = f \circ p(z_1, \theta) = f(z_1, z_2) \) on \( \Gamma_z \). Let \( \sigma_j(z_1, z_2) = \tilde{\sigma}(z_1, \theta) \) be the \( j \)th Cesàro mean for \( \tilde{f} \). It is a straightforward generalization of Fourier series theory on the circle that \( \sigma_j \) converges uniformly to \( \tilde{f} \) on \( X_T \). So, in order to show \( f \in R(X_T) \), we need only to show \( \sigma_j \)'s in \( R(X_T) \) for all \( j \). Fix \( z_j^m \mu \) in \( S \),

\[
\int \sigma_j z^m \, d\mu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_z} \sigma_j(z_1, z_2) z^m \frac{dz_2}{z_2} \right) d\nu(z_1)
\]

As \( j \to \infty \), \( \int \sigma_j z^m \, d\mu \to \int f z^m \, d\mu = 0 \) by hypothesis, while the right hand side approaches \( \int f_{-m}(z_1)(1 - z_1 \bar{z}_1)^{m/2} \, d\nu(z_1) \). So, we get \( \int f_{-m}(z_1)(1 - z_1 \bar{z}_1)^{m/2} \, d\nu(z_1) = 0 \) for all \( \nu \) in \( R(T)^\perp \). Therefore, \( f_{-m}(z_1)(1 - z_1 \bar{z}_1)^{m/2} = h_{-m}(z_1) \) for some \( h_{-m}(z_1) \) in \( R(T) \). And

\[
\sigma_j(z_1, z_2) = \frac{1}{j} \sum_{n=0}^{\infty} \sum_{k=-n}^{n} f_k(z_1) e^{ik\theta}
\]

\[
= \frac{1}{j} \sum_{n=0}^{\infty} \sum_{k=-n}^{n} h_k(z_1)(1 - z_1 \bar{z}_1)^{k/2} e^{ik\theta}
\]

is in \( R(X_T) \). So is \( f \).

We can now assert that the linear span of \( S \) is weak-* dense in \( R(X_T)^\perp \). For, if not, then there exists \( g \) in \( C(X_T) \) such that \( g \) annihilates \( S \) as well as its linear span,
yet $\int g \, dt \neq 0$ for some $\tau \in R(X_T)^\perp$ which is not in the span of $S$. By (\#), $g$ is in $R(X_T)$. Hence $\int g \, dt = 0$, a contradiction. So the linear span of $S$ is weak-* dense in $R(X_T)^\perp$. Q.E.D.

**Proof of Lemma.**

$$\frac{\partial \phi_0}{\partial \xi_1} = \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} \frac{\partial \phi}{\partial \xi_1} \left( -\frac{\xi_1}{1 - \xi_1 \tilde{z}_1} e^u + \frac{\partial \phi}{\partial \xi_2} \left( -\frac{\xi_1}{1 - \xi_1 \tilde{z}_1} e^{-u} \right) \right) \right] \frac{dt}{2\pi}.$$ (4)

On the other hand,

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \xi_2} \frac{\partial \xi_2}{\partial t} + \frac{\partial \phi}{\partial \xi_2} \frac{\partial \xi_2}{\partial t} = i \frac{\partial \phi}{\partial \xi_2} \left( 1 - \frac{\xi_1 \tilde{z}_1}{1 - \xi_1 \tilde{z}_1} \right) e^u - i \frac{\partial \phi}{\partial \xi_2} \left( 1 - \frac{\xi_1 \tilde{z}_1}{1 - \xi_1 \tilde{z}_1} \right) e^{-u}.$$

So,

$$\frac{\partial \phi}{\partial \xi_2} e^u = \frac{1}{i} \frac{\partial \phi}{\partial t} \frac{1}{1 - \xi_1 \tilde{z}_1} e^{-u} + \frac{\partial \phi}{\partial \xi_2} e^{-u}.$$

Substituting the above into (4), we get

$$\frac{\partial \phi_0}{\partial \xi_1} = \int_{-\pi}^{\pi} \left( \frac{\partial \phi}{\partial \xi_1} + \frac{1}{2i} \frac{\partial \phi}{\partial \xi_2} \frac{\xi_1}{1 - \xi_1 \tilde{z}_1} \frac{\partial \phi}{\partial \xi_2} \frac{1}{1 - \xi_1 \tilde{z}_1} e^{-u} \right) \frac{dt}{2\pi} = \int_{-\pi}^{\pi} \left( \frac{\partial \phi}{\partial \xi_1} - \frac{\partial \phi}{\partial \xi_2} \right) \frac{dt}{2\pi} = \left( \frac{\partial \phi}{\partial \xi_1} - \frac{\partial \phi}{\partial \xi_2} \right) \frac{1}{2\pi}.$$ (4)

The term $j(\partial \phi / \partial t)(dt/2\pi) = 0$, since, on $\Gamma_{\xi_1}$, $\partial \phi / \partial t = \sum_{n=-\infty}^{\infty} (in) \phi_n e^{int}$ has no constant term. Q.E.D.

It is an immediate consequence of (\#) that we have

**Theorem 2.** Let $\phi$ be a smooth function with $\partial \phi / \partial z_i \equiv 0$ on $X_T$, $i = 1, 2$. Then $\phi \in R(X_T)$.

**Proof.** Since $\phi$ is annihilated by all elements of $S$ so by (\#) $\phi \in R(X_T)$. Q.E.D.

Next, we state a theorem about tame measures in general which is derived from the proof of a theorem due to Weinstock [2, Theorem 1.4].

**Theorem (Weinstock).** Let $X$ be a compact subset of $C^n$. If $\mu$ is a tame measure on $X$ and $\{U_a\}_{i=1}^{N}$ is a finite open covering of $X$, then there exist $\mu_a$ orthogonal measures for $R(X \cap \bar{U_a})$, where each $\mu_a$ has its support contained in $X \cap U_a$, and $\mu = \sum_{a=1}^{N} \mu_a$. 

Proof. Let \(\{\sigma_i\}_{i=1}^n\) be measures supported on \(X\), such that \(\mu = -\sum_{i=1}^n \frac{\partial \sigma_i}{\partial z_i}\). Let \(\{\phi_a\}\) be a smooth partition of unity subordinate to \(\{U_a\}\) satisfying

(i) \(0 < \phi_a < 1\) and \(\text{supp} \phi_a \subset U_a\),

(ii) \(\sum_{a=1}^N \phi_a = 1\).

Then,

\[
\mu = -\sum_{i=1}^n \frac{\partial}{\partial z_i} \left( \sum_{a=1}^N \phi_a \sigma_i \right) = -\sum_{i=1}^n \sum_a \frac{\partial \phi_a}{\partial z_i} \sigma_i - \sum_{a=1}^N \sum_i \phi_a \frac{\partial \sigma_i}{\partial z_i} = -\sum_a \sum_i \frac{\partial \phi_a}{\partial z_i} \sigma_i - \sum_a \phi_a \left( -\sum \frac{\partial \phi_a}{\partial z_i} \sigma_i + \phi_a \mu \right) = \sum \mu_a, \quad \text{where } \mu_a = -\sum_{i=1}^n \frac{\partial \phi_a}{\partial z_i} \sigma_i + \phi_a \mu.
\]

To show that \(\mu_a \perp R(X \cap \overline{U}_a)\), for any \(g \in R_0(X \cap \overline{U}_a)\),

\[
\int g \, d\mu_a = -\sum_{i=1}^n \int g \frac{\partial \phi_a}{\partial z_i} \, d\sigma_i + \int g \phi_a \, d\mu = -\sum_{i=1}^n \int g \frac{\partial \phi_a}{\partial z_i} \, d\sigma_i + \sum_{i=1}^n \int \frac{\partial (g \phi_a)}{\partial z_i} \, d\sigma_i = -\sum_{i=1}^n \int g \frac{\partial \phi_a}{\partial z_i} \, d\sigma_i + \sum_{i=1}^n \int g \frac{\partial \phi_a}{\partial z_i} \, d\sigma_i \quad \text{since } \frac{\partial g}{\partial z_i} = 0 \forall i
\]

\[= 0.
\]

So \(\mu_a\) annihilates \(R_0(X \cap \overline{U}_a)\), hence will annihilate its closure \(R(X \cap \overline{U}_a)\). Since \(\text{supp} \mu_a \subset \text{supp} \phi_a \subset \text{supp} \mu\); we have that \(\text{supp} \mu_a \subset X \cap U_a\) and the theorem is proved. Q.E.D.

With the help of (\#) and the above theorem we can now assert that \(R(X_T)\) is a local algebra.

Theorem 3. Let \(\{U_a\}\) be a finite open covering of \(X_T\). Let \(f \in C(X_T)\) be such that the restriction of \(f\) to \(X_T \cap U_a\) is in \(R(X_T \cap U_a)\) for all \(a\). Then \(f\) is in \(R(X_T)\).

Proof. Let \(S\) be as in Theorem 1. For any \(\mu \in S\), \(\mu\) is tame by Theorem 1. It follows from the above theorem that there exist \(\mu_a\)'s such that \(\text{supp} \mu_a \subset X_T \cap U_a\), \(\mu_a \perp R(X_T \cap \overline{U}_a)\) and \(\sum \mu_a = \mu\). Hence, by hypothesis,

\[
\int f \, d\mu = \sum_{a} \int_{X_T \cap U_a} f \, d\mu_a = 0.
\]

\(f\) is annihilated by \(S\), and \(f\) is then in \(R(X_T)\) by (\#). Q.E.D.
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