TAYLOR-DIRICHLET SERIES AND ALGEBRAIC DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. It is proved that if a convergent Taylor-Dirichlet series
\[ \sum_{k=0}^{\infty} P_k(s)e^{-\lambda_k t}, \quad s = \sigma + it, \lambda_k \in \mathbb{C}, P_k(s) \in \mathbb{C}[s], \Re(\lambda_k) > \infty, \]
satisfies an algebraic differential-difference equation then the set of its exponents \( \{\lambda_k\}_{k=0}^\infty \) has a finite, linear, integral basis. This generalizes a theorem of A. Ostrowski.

An application of the theorem to a problem of oscillation in neuro-muscular systems is indicated.

1. Introduction. According to a well-known theorem of A. Ostrowski [7, Satz 6, p. 260] if a convergent Dirichlet series
\[ \sum_{k=0}^{\infty} a_k e^{-\lambda_k t}, \quad s = \sigma + it, a_k \in \mathbb{C}, \]
with real exponents \( \lambda_0 < \lambda_1 < \ldots \to \infty \) satisfies an algebraic differential-difference equation then the set of its exponents \( \{\lambda_k\}_{k=0}^\infty \) has a finite, linear, integral basis.

In this paper we show that Ostrowski's theorem continues to hold for a convergent Taylor-Dirichlet series
\[ \phi(s) = \sum_{k=0}^{\infty} P_k(s)e^{-\lambda_k t}, \quad \lambda_k \in \mathbb{C}, P_k(s) \not\equiv 0, \tag{1} \]
where the \( P_k \) are polynomials with complex coefficients and the real parts \( \lambda_k \) of \( \lambda_k \) satisfy \( \lambda_0 < \lambda_1 < \ldots \to \infty \). The theorem is stated and proved in §3. In §4 we briefly indicate an area of application to neuro-muscular systems.

We recall that an algebraic differential-difference equation has the form
\[ G(x, f^{(m_1)}(x + h_1), \ldots, f^{(m_r)}(x + h_r)) = 0 \tag{2} \]
where \( G \) is a polynomial
\[ G(x, x_1, \ldots, x_r) = \sum C_{k_1, \ldots, k_r} x_1^{k_1} \cdots x_r^{k_r} \]
with coefficients depending on \( x \), and the ordered pairs \( (h_1, m_1), \ldots, (h_r, m_r) \) of real numbers \( h_k \) and nonnegative integers \( m_k \) are distinct. By the total degree of \( G \) is meant, as usual, the maximum value of the sums \( k_1 + \cdots + k_r \). For a discussion
of convergence of (1) refer to Valiron [10] where it is shown that if the degrees $\mu_k$ of $P_k(s)$ and the exponents $\lambda_k$ satisfy the conditions

$$\lim_{k \to \infty} \frac{\mu_k}{\lambda_k} = 0, \quad \lim_{k \to \infty} \frac{\log k}{\lambda_k} = 0$$

then the region of convergence of (1) is the same as that of the associated classical Dirichlet series $\sum_{k=1}^{\infty} A_k \exp(-s\lambda_k)$, where $A_k$ is the maximum of the moduli of the coefficients of $P_k$. Convergence of these latter series has been treated by a number of authors, for instance, Väisälä [9] and Miškelevičiūs [5]. Further results on Taylor-Dirichlet series can be found in Lepson [3], Lunc [4] and Blambert and Berland [1].

2. Lemmas. The proof of Ostrowski’s theorem depends on three lemmas, the first of which [7, p. 246] states that an analytic function $g(s)$ which satisfies an algebraic differential-difference equation of the form (2) also satisfies an equation

$$F(f'(s + h_1), \ldots, f'(s + h_r)) = 0$$

where the polynomial $F$ does not contain the variable $s$ as one of its arguments and where the total degree of $F$ does not exceed that of $G$. This result carries over without change to (1). Lemmas 1 and 2 which follow are modifications, respectively, of Ostrowski’s Hilfssatz 2, p. 247 and Hilfssatz 3, p. 248 of [7].

**Lemma 1.** The exponential polynomial

$$E(\lambda) = \sum_{k=1}^{q} P_k(\lambda) e^{\alpha_k \lambda} \neq 0, \quad P_k(\lambda) \in \mathbb{C}[\lambda],$$

where the exponents $\alpha_k$ are distinct real numbers has some zero-free right half-plane; i.e. there exists a positive number $B$ such that all zeros of $E(\lambda)$ have real part less than $B$.

**Proof.** Well known; see, e.g., Langer [2, p. 224].

**Definition 1.** A Taylor-Dirichlet series (1) is said to satisfy the algebraic differential-difference equation (3) formally if all coefficients of the series arising from formal substitution of (1) in the left-hand side of (3) vanish identically.

**Definition 2.** Throughout, the ‘exponent’ of a term of the form $P(s)e^{-\lambda s}$ will refer to the number $\lambda$.

We now state and prove the main lemma.

**Lemma 2.** If the Taylor-Dirichlet series (1) satisfies an algebraic differential-difference equation formally in which the variable $s$ does not occur explicitly then every exponent $\lambda_k$ of sufficiently large index $k$ can be expressed as a linear, integral combination of the preceding exponents $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$.

**Proof.** We prove the lemma for $0 < \lambda_0$ since the general situation is treated exactly as in [7]. By hypothesis, $\phi(s)$ satisfies an algebraic differential-difference equation of the form (3) formally. Let $N$ be the total degree of the polynomial $F$ and suppose that $\phi(s)$ does not satisfy formally any nontrivial algebraic differential-difference equation of total degree less than $N$. 

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The partial derivatives

\[ F_p\left(f^{(m_v)}(s + h_r)\right) = \frac{\partial F\left(f^{(m_v)}(s + h_r)\right)}{\partial \left(f^{(m_v)}(s + h_r)\right)} \]

where we have written \( F(f^{(m_v)}(s + h_r)) \) in place of \( F(f^{(m_v)}(s + h_r), \ldots, f^{(m_v)}(s + h_r)) \) are clearly polynomials in \( f^{(m_v)}(s + h_r) \), \( v = 1, 2, \ldots, r \), of total degree less than \( N \).

By the minimality of \( N \), the expression \( F_p(\phi^{(m_v)}(s + h_r)) \) does not vanish formally. We can write the first (ordered by real parts of exponents) nonzero term of this series in the form

\[ Q_p(s)e^{-\Lambda ps} \quad (4) \]

where \( \Lambda_p \) is a linear, integral combination of a finite number of \( \lambda_k \)'s and \( Q_p(s) \) is a polynomial in \( s \) with coefficients depending on these same \( \lambda_k \)'s. Let \( I_0 = \min_{1 < p < r} \{\Lambda_p\} \) and suppose there are \( m \) such terms so that \( \Lambda_{p_1} = \cdots = \Lambda_{p_m} = I_0 \) and that for all other terms of the form (4) we have \( \Lambda_p > I_0 \). Clearly only a finite number of terms of the series \( \phi^{(m_v)}(s + h_r) \) contribute to the leading nonzero term (4) (or to the preceding terms which have cancelled). It follows from this and the monotonicity of the \( \lambda_k \) that a positive number \( l_1 \) exists such that if

\[ \lambda_n > l_1 \quad (5) \]

then \( \phi(s) \) can be split up into two sums

\[ \phi(s) = A_n(s) + R_n(s) \]

where

\[ A_n(s) = \sum_{k=0}^{n-1} P_k(s)e^{-\lambda_k s}, \quad R_n(s) = \sum_{k=n}^{\infty} P_k(s)e^{-\lambda_k s} \]

and the series \( F_p(A^{(m_v)}(s + h_r)) \) will have the same leading nonzero term as \( F_p(\phi^{(m_v)}(s + h_r)) \), namely (4).

Application of Taylor's formula for multivariate polynomials yields the finite expansion

\[ F(\phi^{(m_v)}(s + h_r)) = F(A^{(m_v)}(s + h_r)) + \sum_{1 < p < r} F_p(A^{(m_v)}(s + h_r))R^{(m_v)}(s + h_r) \]

\[ + \sum_{1 < p < \eta < r} F_{p,\eta}(A^{(m_v)}(s + h_r))R^{(m_v)}(s + h_r) \]

\[ + \ldots \quad (6) \]

where \( F_{p,\eta} \) denotes the second-order derivative apart from factorial coefficients.

Consider the third member on the right-hand side of (6) (reserving the word 'term' for series). A lower bound for the real part of the exponent of the first nonzero term of this expression as a formal Taylor-Dirichlet series is clearly \( 2\lambda_n \), with corresponding lower bounds for the fourth, fifth, \ldots members on the right-hand side of (6) being \( 3\lambda_n \), \( 4\lambda_n \), etc.
In order to express $R_n^{(m,\lambda)}(s)$ in a simple form we use the notation $D = d/ds$, the formal differential operator:

$$R_n^{(m,\lambda)}(s) = \frac{d^{(m,\lambda)}}{ds^{m,\lambda}} \left( \sum_{k=n}^{\infty} P_k(s)e^{-\lambda s} \right) = e^{-\lambda s}(D - \lambda_n)^{m,\lambda}(P_n(s)) + \ldots.$$ 

We can now write the initial term of the second member on the right-hand side of (6) as

$$e^{-(l_0 + \lambda_n)}E(s, \lambda_n),$$

where

$$E(s, \lambda) = \sum_{j=1}^{m} Q_{p_j}(s)(D - \lambda)^{m,\lambda}(P_{n_j}(s))e^{-\lambda h_j}$$

and is, for fixed $s$, an exponential polynomial of the kind considered in Lemma 1. Since the polynomials $Q_{p_1}, \ldots, Q_{p_m}$ and $P_n$ are $\equiv 0$ and the ordered pairs $(h_{p_1}, m_{p_1}), \ldots, (h_{p_n}, m_{p_n})$ are distinct, there exists (at least) one value of $s$, say $s = s^*$ for which $E(s^*, \lambda) \equiv 0$ as a function of $\lambda$.

We may therefore apply Lemma 1 to $E(s^*, \lambda)$ and infer the existence of a positive number $l_2$ such that for all $\lambda_n$ whose real part $\lambda'_n$ satisfies

$$\lambda'_n > l_2$$

we have $E(s^*, \lambda_n) \neq 0$.

The term (7) therefore cannot cancel with the initial term of the third, fourth, . . . members on the right-hand side of (6) provided $l_0 + \lambda'_n < 2\lambda'_n$; that is, if

$$\lambda'_n > l_0.$$  

(9)

Combining the above we see from (5), (8) and (9) that if we take $\lambda'_n > l_0 + l_1 + l_2$ then (7) holds and the nonvanishing term (4) cannot be cancelled by any of the third, fourth, . . . members on the right-hand side of (6). By hypothesis, however, each term of the series on the right-hand side of (6) must vanish; therefore $F(A_n^{(m,\lambda)}(s + h_j))$ must contain a term whose exponent is $l_0 + \lambda_n$. Hence by the definition of $A_n(s)$, $l_0 + \lambda_n$ must be a linear, integral combination of $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. The same is true for $l_0$ since $\lambda'_n > l_0$, and hence we see that $\lambda_n$ itself is a linear, integral combination of the preceding exponents.

3. Statement and proof of the theorem.

**Theorem.** If the convergent Taylor-Dirichlet series (1) satisfies an algebraic differential-difference equation then the set of its exponents \(\{\lambda_k\}_{k=0}^{\infty}\) has a finite, linear, integral basis.

**Proof.** By the remarks at the beginning of §2 it suffices to show that if $\phi(s)$ (defined by (1)) satisfies (3) then it also satisfies this equation formally, for then we may use Lemma 2 to complete the proof. In order to obtain the desired contradiction we therefore assume that $\phi(s)$ does not satisfy (3) formally. We can write the first (ordered, as before, according to real parts of exponents) nonzero term of the series obtained by formal substitution of $\phi(s + h_j)$ into the left-hand side of (3) as
where $\Lambda$ is a linear combination of a finite number of $\lambda_k$'s and $Q(s)$ is a polynomial in $s$ whose coefficients depend on these $\lambda_k$'s. We obviously have

\[ Q(s) \equiv 0. \quad (11) \]

We proceed to split $\phi(s)$ into two sums as in the proof of Lemma 2, $\phi(s) = A_n(s) + R_n(s)$ where $n$ is taken so large that $\lambda'_n > \Lambda'$, thus ensuring that the first nonzero term of $F(A_n^{(m)}(s + h_\nu))$ will also be $Q(s)e^{-\Lambda s}$.

Writing

\[ S_n(s) = P_n(s) + P_{n+1}(s)e^{-(\lambda_{n+1} - \lambda_n)s} + \ldots \]

then

\[ \phi(s) = A_n(s) + e^{-\lambda s}S_n(s) \quad (12) \]

and

\[ S_n(s) = O(s^{\mu_n}), \quad s \to \infty, \]

where $\mu_n = \deg P_n(s)$.

Here and subsequently all limits are taken as $s$ assumes real values. From (12) we get

\[ \phi^{(m)}(s + h_\nu) = A_n^{(m)}(s + h_\nu) + e^{-\lambda s}T_n^{(m)}(s, \lambda_n) \quad (13) \]

where

\[ T_n^{(m)}(s, \lambda_n) = O(s^{\mu_n}), \quad s \to \infty. \quad (14) \]

Replacing (12) and (13) into the left-hand side of (3) and applying Taylor's formula we have

\[ 0 = F(A_n^{m}(s + h_\nu)) = F(A_n^{(m)}(s + h_\nu)) + e^{-\lambda s}\Phi(s), \quad (15) \]

where

\[ \Phi(s) = O(s^M), \quad s \to \infty, \quad (16) \]

for some positive integer $M$.

On the other hand we can write

\[ F(A_n^{(m)}(s + h_\nu)) = e^{-\Lambda t}(Q(s) + h(s)) \quad (17) \]

with $h(s) \to 0$ as $s \to \infty$.

Thus if we insert the right-hand side of (17) in (15) we get

\[ 0 = e^{-\Lambda t}(Q(s) + h(s)) + e^{-\lambda s}\Phi(s). \quad (18) \]

Multiplying (18) by $e^{\Lambda s}$ and taking the limit as $s \to \infty$ we obtain $Q(s) \equiv 0$ which contradicts (11) and proves the theorem.

4. An application. We indicate briefly how our theorem may be applied to a solution of the following algebraic differential-difference equation

\[ \sum_{k=0}^{4} a_k f^{(k)}(t) + \sum_{k=1}^{3} \{ b_k f(t - t_k) + c_k f^{(1)}(t - t_k) \} = 0 \quad (19) \]
occurring in the investigation of oscillations in neuro-muscular systems [6]. Here we write \( f^{(0)}(t) \) for \( f(t) \) and \( f^{(1)}(t) \) for \( f'(t) \). The \( a_k, b_k, c_k \) denote complex constants and \( 0 < t_1 < t_2 < t_3 \).

Substituting \( f(t) = e^{st} \) in (19) gives

\[
e^{st} \sum_{k=0}^{4} a_k s^k + \sum_{k=1}^{3} (b_k + c_k s)e^{(t_3 - s)t} = 0
\]

for the transcendental characteristic equation. By a well-known theorem of Pontryagin [8] (20) has in general infinitely many roots \( s \) with negative real part. There are, moreover, only finitely many multiple roots.

Denote the roots of (20) having negative real part by

\[
s_k = -\rho_k + iq_k, \quad \rho_k > 0, \quad k = 1, 2, \ldots,
\]

where \( s_k \) has multiplicity \( n_k \).

The corresponding particular solutions of (19) are then

\[
t^\nu e^{-\rho_k t} \cos q_k t, \quad t^\nu e^{-\rho_k t} \sin q_k t, \quad \nu = 0, 1, \ldots, n_k - 1.
\]

The general solution of (20) has the form

\[
f(t) = \sum_{k=1}^{\infty} e^{-\rho_k t} \{ P_k(t) \cos q_k t + Q_k(t) \sin q_k t \}
\]

where \( P_k(t) \) and \( Q_k(t) \) are polynomials in \( t \) of degree \( n_k - 1 \) with complex coefficients such that (21) converges absolutely and uniformly for \( t > 0 \) and is four times differentiable.

Applying our theorem to the function \( f(t) \) there exists then a finite set of numbers \( p_1, p_2, \ldots, p_n \) such that

\[
f(t) = \sum_{k=1}^{\infty} e^{-\Lambda_k t} \{ P_k(t) \cos q_k t + Q_k(t) \sin q_k t \}
\]

where \( \Lambda_k = \sum_{j=1}^{n} \alpha_{kj} p_j \), for some integers \( \alpha_{kj} \).

The application described above has served primarily as an example to illustrate the possible usefulness of the theorem. It is intended to pursue further such applications in a subsequent paper.

References


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