LINEAR DIFFERENTIAL EQUATIONS
WITH INTERVAL SPECTRUM

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Abstract. Let $E$ be the set of equations in the hull of a fixed two-dimensional, almost-periodic, linear differential equation with interval spectrum which admit a bounded solution. Among other things, we prove that $E$ is of first category and of measure zero.

1. Introduction. In [8], [9], Sacker and Sell defined the spectrum of a linear, nonautonomous differential system $\dot{x} = B(t)x$. In [2], it was proved that a two-dimensional example of Millionsčikov [4] with almost-periodic coefficients has interval spectrum. Our purpose here is to prove several propositions concerning two-dimensional linear systems with uniquely ergodic hull (2.1 and 2.2 below) which have interval spectrum. In particular, we prove a result which implies the following statement. Let $\Omega$ denote the hull of the system, let $\mu_0$ be the unique ergodic measure on $\Omega$, and let the spectrum of the system be $[-\beta, \beta]$ for some $\beta > 0$ (the spectrum may be made symmetric about zero by a simple normalization). Let $\Omega_0 = \{\omega \in \Omega: \text{the equation defined by } w \text{ (see 2.1 and 2.3) admits a bounded solution}\}$. By the definition of spectrum and [8, Theorem 3], $\Omega_0$ is nonempty; we prove that $\mu_0(\Omega_0) = 0$, and that $\Omega_0$ is of first category in $\Omega$.

2. Preliminaries. We introduce notation and review some definitions and results.

2.1. Definitions. Let $C$ be the space of all continuous mappings from $\mathbb{R}$ to the set of $2 \times 2$ real matrices. Give $C$ the topology of uniform convergence on compact sets. The map $\Phi: C \times \mathbb{R} \to C: (A, t) \mapsto A_t$, where $A_t(s) = A(t + s)$, defines a real flow [1] on $C$. Suppose $B \in C$ is uniformly bounded and uniformly continuous. Then $\Omega = \overline{\{B_t: t \in \mathbb{R}\}} \subset C$ is compact metric, and $\Phi_{|_C \times \mathbb{R}}$ defines a flow $(\Omega, \mathbb{R})$. We can "extend $B$ to $\Omega$" as follows: let $b(\omega) = \omega(0)$ ($\omega \in \Omega$); then $b(\omega_t) = \omega_t(0) = \omega(t)$ ($\omega \in \Omega$, $t \in \mathbb{R}$). In particular, if $\omega_0 \equiv B \in \Omega$, then $b(\omega_0 \cdot t) = B(t)$. We call $\Omega$ the hull of $B$.

2.2. Definitions, notation. Let $(X, \mathbb{R})$ be a flow, where $X$ is a compact metric space. We denote the "position" of $x \in X$ after "time" $t \in \mathbb{R}$ by $x \cdot t$. Say $(X, \mathbb{R})$ is minimal if the orbit $\{x \cdot t: t \in \mathbb{R}\}$ is dense in $X$ for all $x \in X$. Say $(X, \mathbb{R})$ is uniquely ergodic if there is exactly one measure $\mu$ on $X$ which is invariant with respect to $(X, \mathbb{R})$. Here and below, "measure" means "Radon probability measure". See [1], [7].

2.3. Notation. Let $B, \Omega, b$ be as in 2.1 (thus $B$ is a uniformly bounded and uniformly continuous, $2 \times 2$-matrix valued function on $\mathbb{R}$).

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Consider the ODEs

\[ \dot{x} = B(t)x, \]  
\[ \dot{x} = b(\omega \cdot t)x \quad (\omega \in \Omega). \]

We say that equation (1) "induces" equations \((1)_\omega\).

2.4. Definitions. In a well-known way, equations \((1)_\omega\) induce a flow (a \textit{linear-skew-product flow}, or LSPF) on \(\Omega \times \mathbb{R}^2\) \([8], [10]\). Let \(L\) denote this LSPF. The \textit{spectrum} of \(L\), \(\text{Sp}(L)\), is defined to be \(\{\lambda \in \mathbb{R} : \text{the equation } \dot{x} = (-\lambda I + b(\omega \cdot t))x \text{ admits a nonzero bounded solution for some } \omega \in \Omega\}\); see \([8]\).

As a consequence of the Sacker-Sell spectral theorem \([9, \text{Theorem 2}]\), and techniques of \([2]\), we have the following.

2.5. Proposition. Let \(L\) be the LSPF of 2.4. Suppose \((\Omega, \mathbb{R})\) is uniquely ergodic. Then \(\text{Sp}(L)\) is (i) a single point or (ii) two points or (iii) a nondegenerate closed interval.

2.6. Definitions. Let \(L\) be the LSPF of 2.4. Let \(K \subset \mathbb{R}^2\) be the unit circle, and let \(P\) be projective 1-space. Since \(L\) is linear, it induces flows on \(\Omega \times K\) and on \(\Sigma = \Omega \times P\); these are defined as follows. If \((\omega, x_0) \in \Omega \times K\), and if \(x(t)\) is the solution to \((1)_\omega\) such that \(x(0) = x_0\), then \((\omega, x_0) \cdot t = (\omega \cdot t, x(t)/||x(t)||)\). If \((\omega, l) \in \Sigma\) (thus \(l\) is a line through the origin in \(\mathbb{R}^2\)), and if \(x_0\) is a nonzero vector in \(l\), then \((\omega, l) \cdot t = (\omega \cdot t, l(t))\), where \(l(t)\) is the line containing \(x(t)\), and \(x(t)\) solves \((1)_\omega\) with \(x(0) = x_0\). Let \(\gamma : \Sigma = \Omega \times P \to \Omega\) be the projection onto the first factor.

2.7. Definitions. Let \(\theta\) be the (usual) angular coordinate on \(K\), and let \(\varphi = 2\theta\) be the corresponding coordinate on \(P^1\). Let us write

\[ b(\omega) = \begin{bmatrix} a(\omega) & -c(\omega) \\ c(\omega) & a(\omega) \end{bmatrix} + \begin{bmatrix} \delta(\omega) & \varepsilon(\omega) \\ \varepsilon(\omega) & -\delta(\omega) \end{bmatrix}. \]

In polar coordinates \((r, \theta)\), equations \((1)_\omega\) become

\[ \dot{\theta} = c(\omega \cdot t) + \varepsilon(\omega \cdot t)\cos 2\theta(t) - \delta(\omega \cdot t)\sin 2\theta(t), \]
\[ \frac{d}{dt} \ln r(t) = a(\omega \cdot t) + \delta(\omega \cdot t)\cos 2\theta(t) + \varepsilon(\omega \cdot t)\sin 2\theta(t). \]

From \((2)_\omega\), the flow \((\Sigma, \mathbb{R})\) is independent of \(a(\omega) = \frac{1}{2} \text{tr } b(\omega)\). Since our results will depend only on properties of this flow, we will assume \(\text{tr } b(\omega) \equiv 0\).

2.8. Remark. Assume \(\text{tr } b(\omega) \equiv 0\), and define

\[ f_* : \Sigma \to \mathbb{R} : f_*(\omega, \varphi) = \delta(\omega)\cos \varphi + \varepsilon(\omega)\sin \varphi. \]

By \((3)_\omega\) and \((4)\), we have

\[ \frac{1}{t} \ln \frac{||x(t)||}{||x(0)||} = \frac{1}{t} \int_0^t f_*((\omega, \varphi) \cdot s) \, ds \]

whenever \(x(t)\) is a solution to \((1)_\omega\) such that the ray defined by \(x(0) \neq 0\) has coordinate \(\varphi\).

From 2.5 and techniques and results of \([2, \S 3]\), we have...
2.9. Theorem. Suppose $(\Omega, \mathbb{R})$ is uniquely ergodic, and suppose $\text{tr } b(\omega) = 2a(\omega) \equiv 0$. Assume also that $\text{Sp}(L)$ is a nondegenerate interval. Then $\text{Sp}(L) = [-\beta, \beta]$ for some $\beta > 0$. There are exactly two ergodic measures, $\mu_1$ and $\mu_2$ on $\Sigma$, and $\int_{\Sigma} f_* d\mu_1 = -\beta$, $\int_{\Sigma} f_* d\mu_2 = \beta$ (or vice versa). There are disjoint Borel sets $B_i, B_2 \subset \Sigma$ such that $\mu_i(B) = 1$, and $\text{card}(B_i \cap \pi_p^{-1}(\omega)) = 1$ for $\mu_0$-a.a. $\omega \in \Omega$ ($i = 1, 2$). Here $\mu_0$ is the unique invariant measure on $\Omega$.

3. Results.

3.1. Assumptions, notation. Notation $(B, b, \Omega, \Sigma, L, f_*)$ is as in §2. We assume $\text{tr } b(\omega) \equiv 0$, and that $(\Omega, \mathbb{R})$: (i) is minimal; (ii) admits a unique invariant measure $\mu_0$. We assume $\text{Sp}(L)$ is a nondegenerate interval; thus $\text{Sp}(L) = [-\beta, \beta]$ for some $\beta > 0$.

3.2. Remark. If $\text{tr } b(\omega) = 2a(\omega)$ is not identically zero, and if $\text{Sp}(L)$ is a nondegenerate interval, then $\text{Sp}(L) = [a_0 - \beta, a_0 + \beta]$, where: (i) $a_0 = \int_{\Omega} a(\omega) d\mu_0(\omega)$; (ii) $[-\beta, \beta]$ is the spectrum of the LSPF obtained from the equations $x = [b(u \cdot t) - a(u \cdot t)]x$.

The proposition below has been stated in the literature [9]; we give a proof.

3.3. Proposition. The flow $(\Sigma, \mathbb{R})$ contains a unique minimal subflow $(M, \mathbb{R})$.

Proof. The flow $(\Sigma, \mathbb{R})$ contains at least one minimal subflow [1]. Suppose $M_1$ and $M_2$ are disjoint minimal subsets of $\Sigma$. Let $\mu_1$ and $\mu_2$ be the two measures on $\Sigma$ which are ergodic with respect to $(\Sigma, \mathbb{R})$ (2.9). Since each $(M_i, \mathbb{R})$ admits an ergodic measure [7], we must have, say, $\mu_1(M_i) = 1$ ($i = 1, 2$). Then $(M_1, \mathbb{R})$ and $(M_2, \mathbb{R})$ are uniquely ergodic, hence by 2.9 and [7, pp. 498–511], we have

$$\lim_{|t| \to \infty} \frac{1}{t} \int_{0}^{t} f_*(((\omega, \varphi) \cdot s)) \, ds = -\beta \quad \text{resp. } \beta \quad (*)$$

for all $(\omega, \varphi) \in M_1$ resp. $M_2$.

Recall now that $\pi_p: \Sigma \to \Omega$ is the projection. Since $(\Omega, \mathbb{R})$ is minimal, $\pi_p(M_1) = \Omega = \pi_p(M_2)$. It follows that zero cannot be in the spectrum of $L$. For, if it were, some equation $(1)_{\omega_0}$ would have a nonzero bounded solution $x_0(t)$ (2.4). Let $(\omega_0, \varphi_1) \in M_1$, and let $\tilde{x}_1 \neq 0$ be on the line in $\mathbb{R}^2$ defined by $\varphi_1 \in P^1$. If $x_1(t)$ satisfies $(1)_{\omega_0}$ with $x_1(0) = \tilde{x}_1$, let $\Psi(t)$ be the fundamental matrix formed from $x_0(t)$ and $x_1(t)$; $x_0(t)$ and $x_1(t)$ are linearly independent because, by $(*)$, $x_1(t)$ cannot be bounded as $t \to \infty$. Since $\text{tr } b(\omega) \equiv 0$, Liouville's formula implies that $\det \Psi(t) = \text{const}$. But, by $(*)$, $\det \Psi(t) \to 0$ as $t \to -\infty$. This contradiction implies that zero is not in $\text{Sp}(L)$.

Now, by 2.9, $\text{Sp}(L)$ must be a point or two points. This contradicts our assumption (3.1). The proof is completed.

3.4. Remark. If we combine the Birkhoff ergodic theorem [7] with 2.9, we see that there is a set $\Omega_1 \subset \Omega$ of $\mu_0$-measure 1 such that, if $\omega \in \Omega_1$, then $(1)_{\omega}$ has two solutions, $x_1(t)$ and $x_2(t)$, satisfying

$$\lim_{|t| \to \infty} \frac{1}{t} \ln \|x_1(t)\| = \beta, \quad \lim_{|t| \to \infty} \frac{1}{t} \ln \|x_2(t)\| = -\beta.$$
Thus, in a measure-theoretic sense, “most” equations \((1)_{\omega}\) have solutions with “regular” asymptotic behavior. This should be contrasted with the following result.

3.5. **Theorem.** There is a residual set \(\Omega_2 \subset \Omega\) (i.e., a set containing a dense \(G_\delta\)) such that, if \(\omega \in \Omega_2\), then \((1)_{\omega}\) has a solution \(x_\omega(t)\) satisfying

\[
\left( \lim_{t \to -\infty}, \lim_{t \to \infty}, \lim_{t \to -\infty}, \lim_{t \to \infty} \right) \frac{1}{|t|} \ln \|x_\omega(t)\| = (\beta, -\beta, \beta, -\beta).
\]

Here \(\beta\) is defined by \(\text{Sp}(L) = [-\beta, \beta]\).

**Proof.** Consider the function \(f_\ast\) of 2.8. Let \(\mu_1\) and \(\mu_2\) be the two ergodic measures on \(\Sigma\). If \(M\) is the unique minimal subset of \(\Sigma\), then \(\mu_1(M) = 1 = \mu_2(M)\). By 2.9, we may assume that \(\int_M f_\ast \, d\mu_1 = \beta > 0\), and that \(\int_M f_\ast \, d\mu_2 = -\beta < 0\). By [3, Theorem 4.3], there is an invariant, residual subset \(M_2 \subset M\) such that, if \((\omega, \varphi) \in M_2\), then

\[
\left( \lim_{t \to -\infty}, \lim_{t \to \infty}, \lim_{t \to -\infty}, \lim_{t \to \infty} \right) \frac{1}{|t|} \int_0^t f_\ast((\omega, \varphi) \cdot s) \, ds = (\beta, -\beta, \beta, -\beta).
\]

By (5), each such \((\omega, \varphi)\) gives rise to a solution \(x_\omega(t)\) with the properties described in 3.5.

To complete the proof, note that [11, Lemma 3.1] informs us that, if \(\Omega_2 = \pi_p(M_2)\), then \(\Omega_2\) is a residual subset of \(\Omega\).

3.6. **Remarks.** (a) The previous result extends (in the two-dimensional case) Theorem 5 of [9], according to which residually many equations \((1)_{\omega}\) admit solutions \(x(t)\) satisfying \((\lim_{t \to -\infty}, \lim_{t \to \infty}) (1/|t|) \ln \|x(t)\| = \beta\). Our Theorem 3.5 appears to follow from neither the statement nor the proof of this result of [9].

(b) According to results of Millionsčikov [6], every nonzero solution of \((1)_{\omega}\) \((\omega \in \Omega)\) satisfies

\[
\left( \lim_{|t-s| \to \infty}, \lim_{|t-s| \to \infty} \right) \frac{1}{|t-s|} \ln \|x(t)\| = (\beta, -\beta).
\]

This can be proved using 3.5.

Our final theorem implies the result discussed in the Introduction.

3.7. **Theorem.** Given \(\alpha \in \text{Sp}(L) = [-\beta, \beta]\), let \(\Omega_\alpha = \{\omega \in \Omega: \text{equation } (1)_{\omega} \text{ admits a solution } x_\alpha(t) \text{ such that } e^{-|t|} \|x_\alpha(t)\| \text{ is bounded on } -\infty < t < \infty\}\). By 2.4, \(\Omega_\alpha \neq \emptyset\) \((\alpha \in [-\beta, \beta])\).

(a) The set \(\Omega \sim \bigcup_{\alpha \in (-\beta, \beta)} \Omega_\alpha\) contains a residual subset of \(\Omega\), and has \(\mu_\Omega\)-measure 1.

(b) The set \(\Omega \sim \bigcup_{\alpha \in [-\beta, \beta]} \Omega_\alpha\) contains a residual set in \(\Omega\).

**Proof.** (a) Let \(\Omega_1 \subset \Omega\) be the set discussed in 3.4, and let \(\Omega_2 \subset \Omega\) be the set discussed in 3.5. Then \(\mu_\Omega(\Omega_1) = 1\), and \(\Omega_2\) is a residual subset of \(\Omega\). Choose \(\alpha\) such that \(-\beta < \alpha < \beta\). Then \(\Omega_\alpha \cap \Omega_1 = \emptyset\), and \(\Omega_\alpha \cap \Omega_2 = \emptyset\). For, suppose, e.g., that \(\omega \in \Omega_\alpha \cap \Omega_1\). Let \(x_1(t)\) be a solution of \((1)_{\omega}\) satisfying \(\lim_{|t| \to \infty} (1/|t|) \ln \|x_1(t)\| = \beta\), and let \(x_\alpha(t)\) be a solution of \((1)_{\omega}\) satisfying \(e^{-|t|} \|x_\alpha(t)\| < C < \infty\) for all \(t\). If \(\Psi(t)\) is the fundamental matrix solution of \((1)_{\omega}\) whose columns are \(x_1(t)\) and \(x_\alpha(t)\), then
lim_{t \to -\infty} \det \Psi(t) = 0. However, \( \text{tr } b(\omega) = 0 \) and Liouville’s formula imply that 
\( \det \Psi(t) = \text{const} \neq 0 \). Hence \( \Omega_\alpha \cap \Omega_1 \) must be empty. Similarly, \( \Omega_\alpha \cap \Omega_2 = \emptyset \).

(b) We will show only that \( \Omega \sim \Omega_\beta \) contains a residual set; the proof for 
\( \Omega \sim \Omega_{-\beta} \) is similar. By the proof of [3, Theorem 4.3], the minimal subset \( M \) of \( \Sigma \) contains a residual subset \( M_3 \) such that 
\[
(\omega, \varphi) \in M_3 \Rightarrow \left( \lim_{t \to -\infty} \frac{\beta t}{1}, \lim_{t \to -\infty} \int_0^t \left( (\omega, \varphi) \cdot s \right) ds \right) = (-\infty, \infty).
\]

By [11, Lemma 3.1], \( \tau(M_3) = \Omega_3 \) is a residual subset of \( \Omega \). Let \( (\omega, \varphi) \in M_3 \), and let 
\( x(t) \) be a solution to \( (1) \omega \) such that \( x(0) \neq 0 \) lies on the line in \( \mathbb{R}^2 \) determined by \( \varphi \). Then 
\[
\left( \lim_{t \to -\infty} \frac{\beta t}{1}, \lim_{t \to -\infty} e^{-\beta t} \| x(t) \| \right) = (0, \infty).
\]

Now, if \( \omega \) were in \( \Omega_\beta \) as well as in \( \Omega_3 \), then \( (1) \omega \) would have a solution \( x_\beta(t) \) satisfying 
\( \| x_\beta(t) \| \leq C e^{-\beta t} \) for some constant \( C \) \( (-\infty < t < \infty) \). Then \( x(t) \) and 
\( x_\beta(t) \) would be linearly independent. A fundamental matrix argument like that used 
in part (a) would yield a contradiction. We conclude that \( \Omega \sim \Omega_\beta \) contains the 
residual set \( \Omega_3 \).

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