PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, September 1980

LINEAR DIFFERENTIAL EQUATIONS WITH INTERVAL SPECTRUM

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ABSTRACT. Let E be the set of equations in the hull of a fixed two-dimensional, almost-periodic, linear differential equation with interval spectrum which admit a bounded solution. Among other things, we prove that E is of first category and of measure zero.

1. Introduction. In **[8]**, **[9]**, Sacker and Sell defined the *spectrum* of a linear, nonautonomous differential system $\dot{x} = B(t)x$. In **[2]**, it was proved that a two-dimensional example of Millionščikov **[4]** with almost-periodic coefficients has interval spectrum. Our purpose here is to prove several propositions concerning two-dimensional linear systems with uniquely ergodic hull (2.1 and 2.2 below) which have interval spectrum. In particular, we prove a result which implies the following statement. Let Ω denote the hull of the system, let μ_0 be the unique ergodic measure on Ω, and let the spectrum of the system be $[-\beta, \beta]$ for some $\beta > 0$ (the spectrum may be made symmetric about zero by a simple normalization). Let $\Omega_0 = \{\omega \in \Omega$: the equation defined by ω (see 2.1 and 2.3) admits a bounded solution$\}$. By the definition of spectrum and **[8**, Theorem 3], Ω_0 is nonempty; we prove that $\mu_0(\Omega_0) = 0$, and that Ω_0 is of first category in Ω.

2. Preliminaries. We introduce notation and review some definitions and results.

2.1. DEFINITIONS. Let C be the space of all continuous mappings from $\mathbf{R}$ to the set of 2×2 real matrices. Give C the topology of uniform convergence on compact sets. The map Φ: $C \times \mathbf{R} \to C$: $(A, t) \to A_t$, where $A_t(s) = A(t + s)$, defines a real flow **[1]** on C. Suppose $B \in C$ is uniformly bounded and uniformly continuous. Then $\Omega = \text{cls}\{B_t: t \in \mathbf{R}\} \subset C$ is compact metric, and $\Phi|_{\Omega \times \mathbf{R}}$ defines a flow $(\Omega, \mathbf{R})$. We can "extend B to Ω" as follows: let $b(\omega) = \omega(0)$ $(\omega \in \Omega)$; then $b(\omega_t) = \omega_t(0) = \omega(t)$ $(\omega \in \Omega, t \in \mathbf{R})$. In particular, if $\omega_0 \equiv B \in \Omega$, then $b(\omega_0 \cdot t) = B(t)$. We call Ω the *hull* of B.

2.2. DEFINITIONS, NOTATION. Let $(X, \mathbf{R})$ be a flow, where X is a compact metric space. We denote the "position" of $x \in X$ after "time" $t \in \mathbf{R}$ by $x \cdot t$. Say $(X, \mathbf{R})$ is *minimal* if the orbit $\{x \cdot t: t \in \mathbf{R}\}$ is dense in X for all $x \in X$. Say $(X, \mathbf{R})$ is *uniquely ergodic* if there is exactly one measure μ on X which is invariant with respect to $(X, \mathbf{R})$. Here and below, "measure" means "Radon probability measure". See **[1]**, **[7]**.

2.3. NOTATION. Let B, Ω, b be as in 2.1 (thus B is a uniformly bounded and uniformly continuous, 2×2-matrix valued function on $\mathbf{R}$).

Received by the editors August 7, 1979.
AMS (MOS) subject classifications (1970). Primary 34C10, 54H20.

Consider the ODEs

$$\dot{x} = B(t)x, \tag{1}$$

$$\dot{x} = b(\omega \cdot t)x \qquad (\omega \in \Omega). \tag*{$(1)_\omega$}$$

We say that equation (1) "induces" equations $(1)_\omega$.

2.4. Definitions. In a well-known way, equations $(1)_\omega$ induce a flow (a *linear-skew-product flow*, or LSPF) on $\Omega \times \mathbf{R}^2$ **[8]**, **[10]**. Let L denote this LSPF. The *spectrum* of L, $\mathrm{Sp}(L)$, is defined to be $\{\lambda \in \mathbf{R}$: the equation $\dot{x} = (-\lambda I + b(\omega \cdot t))x$ admits a nonzero bounded solution for some $\omega \in \Omega\}$; see **[8]**.

As a consequence of the Sacker-Sell spectral theorem **[9**, Theorem 2], and techniques of **[2]**, we have the following.

2.5. Proposition. *Let L be the LSPF of* 2.4. *Suppose* $(\Omega, \mathbf{R})$ *is uniquely ergodic. Then* $\mathrm{Sp}(L)$ *is* (i) *a single point or* (ii) *two points or* (iii) *a nondegenerate closed interval.*

2.6. Definitions. Let L be the LSPF of 2.4. Let $\mathbf{K} \subset \mathbf{R}^2$ be the unit circle, and let $\mathbf{P}$ be projective 1-space. Since L is linear, it induces flows on $\Omega \times \mathbf{K}$ and on $\Sigma = \Omega \times \mathbf{P}$; these are defined as follows. If $(\omega, x_0) \in \Omega \times \mathbf{K}$, and if $x(t)$ is the solution to $(1)_\omega$ such that $x(0) = x_0$, then $(\omega, x_0) \cdot t = (\omega \cdot t, x(t)/\|x(t)\|)$. If $(\omega, l) \in \Sigma$ (thus l is a line through the origin in $\mathbf{R}^2$), and if x_0 is a nonzero vector in l, then $(\omega, l) \cdot t = (\omega \cdot t, l(t))$, where $l(t)$ is the line containing $x(t)$, and $x(t)$ solves $(1)_\omega$ with $x(0) = x_0$. Let $\pi_p\colon \Sigma = \Omega \times \mathbf{P} \to \Omega$ be the projection onto the first factor.

2.7. Definitions. Let θ be the (usual) angular coordinate on $\mathbf{K}$, and let $\varphi = 2\theta$ be the corresponding coordinate on $\mathbf{P}^1$. Let us write

$$b(\omega) = \begin{bmatrix} a(\omega) & -c(\omega) \\ c(\omega) & a(\omega) \end{bmatrix} + \begin{bmatrix} \delta(\omega) & \varepsilon(\omega) \\ \varepsilon(\omega) & -\delta(\omega) \end{bmatrix}.$$

In polar coordinates (r, θ), equations $(1)_\omega$ become

$$\dot{\theta} = c(\omega \cdot t) + \varepsilon(\omega \cdot t)\cos 2\theta(t) - \delta(\omega \cdot t)\sin 2\theta(t), \tag*{$(2)_\omega$}$$

$$\frac{d}{dt}\ln r(t) = a(\omega \cdot t) + \delta(\omega \cdot t)\cos 2\theta(t) + \varepsilon(\omega \cdot t)\sin 2\theta(t). \tag*{$(3)_\omega$}$$

From $(2)_\omega$, the flow $(\Sigma, \mathbf{R})$ is independent of $a(\omega) = \frac{1}{2}\mathrm{tr}\, b(\omega)$. Since our results will depend only on properties of this flow, we will assume $\mathrm{tr}\, b(\omega) \equiv 0$.

2.8. Remark. Assume $\mathrm{tr}\, b(\omega) \equiv 0$, and define

$$f_*\colon \Sigma \to \mathbf{R}\colon f_*(\omega, \varphi) = \delta(\omega)\cos\varphi + \varepsilon(\omega)\sin\varphi. \tag{4}$$

By $(3)_\omega$ and (4), we have

$$\frac{1}{t}\ln\frac{\|x(t)\|}{\|x(0)\|} = \frac{1}{t}\int_0^t f_*((\omega, \varphi) \cdot s)\, ds \tag{5}$$

whenever $x(t)$ is a solution to $(1)_\omega$ such that the ray defined by $x(0) \neq 0$ has coordinate φ.

From 2.5 and techniques and results of **[2**, §3], we have

2.9. THEOREM. *Suppose* $(\Omega, \mathbf{R})$ *is uniquely ergodic, and suppose* $\operatorname{tr} b(\omega) = 2a(\omega) \equiv 0$. *Assume also that* $\operatorname{Sp}(L)$ *is a nondegenerate interval. Then* $\operatorname{Sp}(L) = [-\beta, \beta]$ *for some* $\beta > 0$. *There are exactly two ergodic measures,* μ_1 *and* μ_2 *on* Σ, *and* $\int_\Sigma f_* \, d\mu_1 = -\beta$, $\int_\Sigma f_* \, d\mu_2 = \beta$ *(or vice versa). There are disjoint Borel sets* $B_1, B_2 \subset \Sigma$ *such that* $\mu_i(B_i) = 1$, *and* $\operatorname{card}(B_i \cap \pi_p^{-1}(\omega)) = 1$ *for* μ_0*-a.a.* $\omega \in \Omega$ $(i = 1, 2)$. *Here* μ_0 *is the unique invariant measure on* Ω.

3. Results.

3.1. ASSUMPTIONS, NOTATION. Notation $(B, b, \Omega, \Sigma, L, f_*)$ is as in §2. We assume $\operatorname{tr} b(\omega) \equiv 0$, and that $(\Omega, \mathbf{R})$: (i) is minimal; (ii) admits a unique invariant measure μ_0. We assume $\operatorname{Sp}(L)$ is a nondegenerate interval; thus $\operatorname{Sp}(L) = [-\beta, \beta]$ for some $\beta > 0$.

3.2. REMARK. If $\operatorname{tr} b(\omega) = 2a(\omega)$ is not identically zero, and if $\operatorname{Sp}(L)$ is a nondegenerate interval, then $\operatorname{Sp}(L) = [a_0 - \beta, a_0 + \beta]$, where: (i) $a_0 = \int_\Omega a(\omega) \, d\mu_0(\omega)$; (ii) $[-\beta, \beta]$ is the spectrum of the LSPF obtained from the equations $\dot{x} = [b(\omega \cdot t) - a(\omega \cdot t)I]x$.

The proposition below has been stated in the literature [9]; we give a proof.

3.3. PROPOSITION. *The flow* $(\Sigma, \mathbf{R})$ *contains a unique minimal subflow* $(M, \mathbf{R})$.

PROOF. The flow $(\Sigma, \mathbf{R})$ contains at least one minimal subflow [1]. Suppose M_1 and M_2 are disjoint minimal subsets of Σ. Let μ_1 and μ_2 be the two measures on Σ which are ergodic with respect to $(\Sigma, \mathbf{R})$ (2.9). Since each $(M_i, \mathbf{R})$ admits an ergodic measure [7], we must have, say, $\mu_i(M_i) = 1$ $(i = 1, 2)$. Then $(M_1, \mathbf{R})$ and $(M_2, \mathbf{R})$ are uniquely ergodic, hence by 2.9 and [7, pp. 498–511], we have

$$\lim_{|t| \to \infty} \frac{1}{t} \int_0^t f_*((\omega, \varphi) \cdot s) \, ds = -\beta \quad \text{resp. } \beta \tag{$*$}$$

for *all* $(\omega, \varphi) \in M_1$ resp. M_2.

Recall now that $\pi_p \colon \Sigma \to \Omega$ is the projection. Since $(\Omega, \mathbf{R})$ is minimal, $\pi_p(M_1) = \Omega = \pi_p(M_2)$. It follows that zero cannot be in the spectrum of L. For, if it were, some equation $(1)_{\omega_0}$ would have a nonzero bounded solution $x_0(t)$ (2.4). Let $(\omega_0, \varphi_1) \in M_1$, and let $\tilde{x}_1 \neq 0$ be on the line in $\mathbf{R}^2$ defined by $\varphi_1 \in \mathbf{P}^1$. If $x_1(t)$ satisfies $(1)_{\omega_0}$ with $x_1(0) = \tilde{x}_1$, let $\Psi(t)$ be the fundamental matrix formed from $x_0(t)$ and $x_1(t)$ ($x_0(t)$ and $x_1(t)$ are linearly independent because, by $(*)$, $x_1(t)$ cannot be bounded as $t \to \infty$). Since $\operatorname{tr} b(\omega) \equiv 0$, Liouville's formula implies that $\det \Psi(t) = \text{const}$. But, by $(*)$, $\det \Psi(t) \to 0$ as $t \to -\infty$. This contradiction implies that zero is not in $\operatorname{Sp}(L)$.

Now, by 2.9, $\operatorname{Sp}(L)$ must be a point or two points. This contradicts our assumption (3.1). The proof is completed.

3.4. REMARK. If we combine the Birkhoff ergodic theorem [7] with 2.9, we see that there is a set $\Omega_1 \subset \Omega$ of μ_0-measure 1 such that, if $\omega \in \Omega_1$, then $(1)_\omega$ has two solutions, $x_1(t)$ and $x_2(t)$, satisfying

$$\lim_{|t| \to \infty} \frac{1}{t} \ln \|x_1(t)\| = \beta, \qquad \lim_{|t| \to \infty} \frac{1}{t} \ln \|x_2(t)\| = -\beta.$$

Thus, in a measure-theoretic sense, "most" equations $(1)_\omega$ have solutions with "regular" asymptotic behavior. This should be contrasted with the following result.

3.5. THEOREM. *There is a residual set* $\Omega_2 \subset \Omega$ *(i.e., a set containing a dense* G_δ*) such that, if* $\omega \in \Omega_2$, *then* $(1)_\omega$ *has a solution* $x_\omega(t)$ *satisfying*

$$\left(\overline{\lim_{t\to\infty}}, \underline{\lim_{t\to\infty}}, \overline{\lim_{t\to-\infty}}, \underline{\lim_{t\to-\infty}} \right) \frac{1}{|t|} \ln\|x_\omega(t)\| = (\beta, -\beta, \beta, -\beta).$$

Here β *is defined by* $\mathrm{Sp}(L) = [-\beta, \beta]$.

PROOF. Consider the function f_* of 2.8. Let μ_1 and μ_2 be the two ergodic measures on Σ. If M is the unique minimal subset of Σ, then $\mu_1(M) = 1 = \mu_2(M)$. By 2.9, we may assume that $\int_M f_*\, d\mu_1 = \beta > 0$, and that $\int_M f_*\, d\mu_2 = -\beta < 0$. By [3, Theorem 4.3], there is an invariant, residual subset $M_2 \subset M$ such that, if $(\omega, \varphi) \in M_2$, then

$$\left(\overline{\lim_{t\to\infty}}, \underline{\lim_{t\to\infty}}, \overline{\lim_{t\to-\infty}}, \underline{\lim_{t\to-\infty}} \right) \frac{1}{|t|} \int_0^t f_*((\omega, \varphi)\cdot s)\, ds = (\beta, -\beta, \beta, -\beta).$$

By (5), each such (ω, φ) gives rise to a solution $x_\omega(t)$ with the properties described in 3.5.

To complete the proof, note that [11, Lemma 3.1] informs us that, if $\Omega_2 = \pi_p(M_2)$, then Ω_2 is a residual subset of Ω.

3.6. REMARKS. (a) The previous result extends (in the two-dimensional case) Theorem 5 of [9], according to which residually many equations $(1)_\omega$ admit solutions $x(t)$ satisfying $(\overline{\lim}_{t\to\infty}, \overline{\lim}_{t\to-\infty})\,(1/|t|)\ln\|x(t)\| = \beta$. Our Theorem 3.5 appears to follow from neither the statement nor the proof of this result of [9].

(b) According to results of Millionščikov [6], every nonzero solution of $(1)_\omega$ $(\omega \in \Omega)$ satisfies

$$\left(\overline{\lim_{|t-s|\to\infty}}, \underline{\lim_{|t-s|\to\infty}} \right) \frac{1}{|t-s|} \ln \frac{\|x(t)\|}{\|x(s)\|} = (\beta, -\beta).$$

This can be proved using 3.5.

Our final theorem implies the result discussed in the Introduction.

3.7. THEOREM. *Given* $\alpha \in \mathrm{Sp}(L) = [-\beta, \beta]$, *let* $\Omega_\alpha = \{\omega \in \Omega$: *equation* $(1)_\omega$ *admits a solution* $x_\alpha(t)$ *such that* $e^{-\alpha t}\|x_\alpha(t)\|$ *is bounded on* $-\infty < t < \infty\}$. *By* 2.4, $\Omega_\alpha \neq \varnothing$ $(\alpha \in [-\beta, \beta])$.

(a) *The set* $\Omega \sim \bigcup_{\alpha \in (-\beta,\beta)} \Omega_\alpha$ *contains a residual subset of* Ω, *and has* μ_0*-measure* 1.

(b) *The set* $\Omega \sim \bigcup_{\alpha \in [-\beta,\beta]} \Omega_\alpha$ *contains a residual set in* Ω.

PROOF. (a) Let $\Omega_1 \subset \Omega$ be the set discussed in 3.4, and let $\Omega_2 \subset \Omega$ be the set discussed in 3.5. Then $\mu_0(\Omega_1) = 1$, and Ω_2 is a residual subset of Ω. Choose α such that $-\beta < \alpha < \beta$. Then $\Omega_\alpha \cap \Omega_1 = \varnothing$, and $\Omega_\alpha \cap \Omega_2 = \varnothing$. For, suppose, e.g., that $\omega \in \Omega_\alpha \cap \Omega_1$. Let $x_1(t)$ be a solution of $(1)_\omega$ satisfying $\lim_{|t|\to\infty}(1/t)\ln\|x_1(t)\| = \beta$, and let $x_\alpha(t)$ be a solution of $(1)_\omega$ satisfying $e^{-\alpha t}\|x_\alpha(t)\| \leqslant C < \infty$ for all t. If $\Psi(t)$ is the fundamental matrix solution of $(1)_\omega$ whose columns are $x_1(t)$ and $x_\alpha(t)$, then

$\lim_{t\to-\infty}\det\Psi(t)=0$. However, $\operatorname{tr} b(\omega)=0$ and Liouville's formula imply that $\det\Psi(t)=\text{const}\neq 0$. Hence $\Omega_\alpha\cap\Omega_1$ must be empty. Similarly, $\Omega_\alpha\cap\Omega_2=\varnothing$.

(b) We will show only that $\Omega\sim\Omega_\beta$ contains a residual set; the proof for $\Omega\sim\Omega_{-\beta}$ is similar. By the proof of [3, Theorem 4.3], the minimal subset M of Σ contains a residual subset M_3 such that

$$(\omega,\varphi)\in M_3\Rightarrow\left(\varliminf_{t\to\infty},\ \varlimsup_{t\to\infty}\right)\left[\beta t+\int_0^t f_*((\omega,\varphi)\cdot s)\,ds\right]=(-\infty,\infty).$$

By [11, Lemma 3.1], $\pi_p(M_3)=\Omega_3$ is a residual subset of Ω. Let $(\omega,\varphi)\in M_3$, and let $x(t)$ be a solution to $(1)_\omega$ such that $x(0)\neq 0$ lies on the line in $\mathbf{R}^2$ determined by φ. Then

$$\left(\varliminf_{t\to\infty},\ \varlimsup_{t\to\infty}\right)e^{-\beta t}\|x(t)\|=(0,\infty).$$

Now, if ω were in Ω_β as well as in Ω_3, then $(1)_\omega$ would have a solution $x_\beta(t)$ satisfying $\|x_\beta(t)\|\leqslant Ce^{-\beta t}$ for some constant C $(-\infty<t<\infty)$. Then $x(t)$ and $x_\beta(t)$ would be linearly independent. A fundamental matrix argument like that used in part (a) would yield a contradiction. We conclude that $\Omega\sim\Omega_\beta$ contains the residual set Ω_3.

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