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CYCLIC VECTORS OF LAMBERT'S WEIGHTED SHIFTS

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ABSTRACT. Let $B(H)$ denote the Banach algebra of all bounded linear operators on an infinite-dimensional separable complex Hilbert space H, and let l^2 be the Hilbert space of all square-summable complex sequences $x = \{x_0, x_1, x_2, \dots\}$. For an injective operator A in $B(H)$ and a nonzero vector f in H, put $w_m = \|A^m f\|/\|A^{m-1}f\|$, $m = 1, 2, \dots$. The operator $T_{A,f}$ on l^2, defined by $T_{A,f}(x) = \{w_1 x_1, w_2 x_2, \dots\}$, is called a weighted (backward) shift with the weight sequence $\{w_m\}_{m=1}^{\infty}$. This paper is concerned with the investigation of the existence of cyclic vectors of $T_{A,f}$. Also it is shown that if A satisfies certain nice conditions, then every transitive subalgebra of $B(H)$ containing $T_{A,f}$ coincides with $B(H)$.

1. Let H be an infinite-dimensional separable complex Hilbert space with an orthonormal basis $\{e_m\}_{m=0}^{\infty}$ and let $B(H)$ be the Banach algebra of all (bounded linear) operators from H into H. If $\{w_m\}_{m=1}^{\infty}$ is a bounded sequence of nonzero complex numbers, the unique operator T on H defined by $Te_0 = 0$ and $Te_m = w_m e_{m-1}$, $m = 1, 2, \dots$, is called the (unilateral backward) weighed shift with the weight sequence $\{w_m\}_{m=1}^{\infty}$. A concrete realisation of the weighted shift is obtained by considering the Hilbert space l^2 of all square-summable complex sequences $x = \{x_0, x_1, x_2, \dots\}$. The weighted shift T on l^2 appears as

$$T\{x_0, x_1, x_2, \dots\} = \{w_1 x_1, w_2 x_2, \dots\}.$$

A vector $x_0 \in H$ is called a cyclic vector of T if $\bigvee_{n=0}^{\infty}\{T^n x_0\} = H$. The existence of cyclic vectors of weighted shifts has been the subject of investigation by many authors; see, for example, Douglas, Shapiro and Shields **[3]**, **[4]**, Gellar **[6]**, Herrero **[7]**, Deddens, Gellar and Herrero **[2]**, Nikolskiĭ **[11]**, **[12]** and Rabindranathan **[14]**.

We say that an operator A is power-bounded if $\|A^n\| \leqslant \delta$ for all $n = 1, 2, 3, \dots$, where δ is a constant.

2. Let $A \in B(H)$ and suppose that A is injective. For a vector $f \neq 0$ in H, let $T_{A,f}$ be the weighted shift on l^2 with the weight sequence $\{w_m\}_{m=1}^{\infty}$, where

$$w_m = \|A^m f\|/\|A^{m-1}f\|. \tag{1}$$

The subnormality of the weighted shifts $T_{A,f}$ has been recently studied by Lambert **[10]**. In this paper we exhibit the existence of cyclic vectors of such weighted shifts under nice conditions on the operator A.

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We set

$$\Delta_k = \sum_{m>k} \left| \frac{x_m}{x_k} \right|^2, \qquad k = 0, 1, 2, \ldots, \tag{2}$$

and prove the following

THEOREM 1. *Let A be power-bounded and such that for every nonzero vector $f \in H$, $A^n f$ does not converge to 0 as $n \to \infty$. Then any vector $x = \{x_m\}_{m=0}^{\infty}$ in l^2, such that*

$$\begin{aligned} &\text{(i)} \quad x_m \neq 0, m = 0, 1, 2, \ldots, \quad \textit{and} \\ &\text{(ii)} \quad \lim_m \Delta_m = 0, \end{aligned} \tag{$*$}$$

is a cyclic vector of $T_{A,f}$.

PROOF. We first observe that

$$\inf_{n>0} \|A^n f\| = \mu(f) > 0 \quad \text{for all } f \neq 0. \tag{3}$$

In fact, as A is power-bounded, let $\|A^n\| \leqslant \delta$, for all $n = 1, 2, \ldots$. Then $\mu(f) = 0$ implies that there exists, for every $\varepsilon > 0$, an $n_0 = n_0(f, \varepsilon)$ such that $\|A^{n_0} f\| < \varepsilon/\delta$; and hence

$$\begin{aligned} \|A^n f\| = \|A^{n-n_0} A^{n_0} f\| &\leqslant \|A^{n-n_0}\| \, \|A^{n_0} f\| \\ &\leqslant \delta \|A^{n_0} f\| < \varepsilon \quad \text{for } n \geqslant n_0. \end{aligned}$$

This contradicts our hypothesis that $A^n f \nrightarrow 0$.

Let $\{e_m\}_{m=0}^{\infty}$ be the standard orthonormal basis for l^2 and let $M = \bigvee_{n=0}^{\infty} \{T_{A,f}^n x\}$. We first show that $e_0 \in M$. Since

$$T_{A,f}^n x = \{w_1 w_2 \ldots w_n x_n, w_2 w_3 \ldots w_{n+1} x_{n+1}, \ldots\},$$

where $w_n = \|A^n f\| / \|A^{n-1} f\|$, $n = 1, 2, \ldots$, it follows that

$$\begin{aligned} \left\| \frac{T_{A,f}^n x}{w_1 w_2 \ldots w_n x_n} - e_0 \right\|^2 &= \sum_{m=n+1}^{\infty} \left(\frac{w_{m-n+1} \ldots w_m}{w_1 \ldots w_n} \right)^2 \left| \frac{x_m}{x_n} \right|^2 \\ &= \sum_{m=0}^{\infty} \left(\frac{w_{m+2} \ldots w_{m+n+1}}{w_1 \ldots w_n} \right)^2 \left| \frac{x_{m+n+1}}{x_n} \right|^2 \\ &= \sum_{m=0}^{\infty} \frac{\|A^{m+n+1} f\|^2 \|f\|^2}{\|A^{m+1} f\|^2 \|A^n f\|^2} \left| \frac{x_{m+n+1}}{x_n} \right|^2 \\ &\leqslant \frac{\|A^n\|^2 \|f\|^2}{\|A^n f\|^2} \sum_{m>n} \left| \frac{x_m}{x_n} \right|^2 \\ &= \frac{\|A^n\|^2 \|f\|^2}{\|A^n f\|^2} \Delta_n \quad \text{(by (2))} \\ &\leqslant \frac{\delta^2 \|f\|^2}{(\mu(f))^2} \Delta_n \quad \text{(by (3))} \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Thus $e_0 \in M$. This implies that for each $n = 1, 2, 3, \ldots$

$$y_n = T_{A,f}^n x - w_1 w_2 \cdots w_n x_n e_0$$

is in M. Proceeding as above, we get

$$\begin{aligned}
\left\| \frac{y_n}{w_2 w_3 \cdots w_{n+1} x_{n+1}} - e_1 \right\|^2 &= \sum_{m=n+2}^{\infty} \left(\frac{w_{m-n+1} \cdots w_m}{w_2 \cdots w_{n+1}} \right)^2 \left| \frac{x_m}{x_{n+1}} \right|^2 \\
&= \sum_{m=0}^{\infty} \left(\frac{w_{m+3} \cdots w_{m+n+2}}{w_2 \cdots w_{n+1}} \right)^2 \left| \frac{x_{m+n+2}}{x_{n+1}} \right|^2 \\
&= \sum_{m=0}^{\infty} \frac{\|A^{m+n+2}f\|^2 \|Af\|^2}{\|A^{m+2}f\|^2 \|A^{n+1}f\|^2} \left| \frac{x_{m+n+2}}{x_{n+1}} \right|^2 \\
&\leqslant \frac{\|A^n\|^2 \|Af\|^2}{\|A^{n+1}f\|^2} \sum_{m>n+1} \left| \frac{x_m}{x_{n+1}} \right|^2 \\
&\leqslant \frac{\delta^2}{(\mu(f))^2} \|Af\|^2 \Delta_{n+1} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

This gives that $e_1 \in M$. Now it is obvious by induction that $e_m \in M$ for all $m = 0, 1, 2, \ldots,$ and hence x is cyclic for $T_{A,f}$.

An operator A in $B(H)$ is said to belong to the class $C_{1.}$ if it is a contraction (i.e. $\|A\| \leqslant 1$) and $A^n f \not\to 0$ for all $f \neq 0$. The class $C_{1.}$ plays an important role in the study of general contractions [**16**, p. 72]. We have

COROLLARY 2. *Theorem* 1 *holds for all* $A \in C_{1.}$.

In our next theorem, we show that the condition $A^n f \not\to 0$ in Theorem 1 can be relaxed in case A is invertible with a power-bounded inverse.

THEOREM 3. *If A is invertible and both A and A^{-1} are power-bounded, then any vector $x = \{x_m\}_{m=0}^{\infty}$ in l^2 satisfying* (*) *is a cyclic vector of $T_{A,f}$.*

PROOF. Since A and A^{-1} are both power-bounded, let $\|A^n\| \leqslant \delta$ for all $n = 0, \pm 1, \pm 2, \ldots$. Now, as in the proof of Theorem 1, it suffices to observe that

$$\begin{aligned}
\left\| \frac{T_{A,f}^n x}{w_1 w_2 \cdots w_n x_n} - e_0 \right\|^2 &\leqslant \frac{\|A^n\|^2 \|f\|^2}{\|A^n f\|^2} \sum_{m>n} \left| \frac{x_m}{x_n} \right|^2 \\
&= \frac{\|A^n\|^2 \|A^{-n}(A^n f)\|^2}{\|A^n f\|^2} \Delta_n \\
&\leqslant \frac{\|A^n\|^2 \|A^{-n}\|^2 \|A^n f\|^2}{\|A^n f\|^2} \Delta_n \\
&= \|A^n\|^2 \|A^{-n}\|^2 \Delta_n \\
&\leqslant \delta^4 \Delta_n \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

3. A strictly cyclic operator algebra $\mathcal{Q}$ on H is a uniformly closed subalgebra of $B(H)$ such that $\mathcal{Q}f_0 = H$ for some vector f_0 in H. In this case f_0 is called a strictly cyclic vector for $\mathcal{Q}$. Moreover, if $Af_0 = 0$, $A \in \mathcal{Q}$ implies that $A = 0$, we say that f_0 is a separating vector for $\mathcal{Q}$. The following lemma is due to Embry [5]:

LEMMA E. *Let f_0 be a strictly cyclic, separating vector for $\mathcal{Q}$. Then there exists a constant K such that $\|A\| \leqslant K\|Af_0\|$ for every A in $\mathcal{Q}$.*

THEOREM 4. *Let $\mathcal{Q}$ be a strictly cyclic operator algebra with a strictly cyclic separating vector f_0, and let $A \in \mathcal{Q}$. Then any vector $x = \{x_m\}_{m=0}^{\infty}$ in l^2 for which* (*) *holds is a cyclic vector for T_{A,f_0}.*

For the proof, we see that

$$\begin{aligned}\left\| \frac{T^n_{A,f_0}x}{w_1 w_2 \cdots w_n x_n} - e_0 \right\|^2 &\leqslant \frac{\|A^n\|^2 \|f_0\|^2}{\|A^n f_0\|^2} \sum_{m>n} \left| \frac{x_m}{x_n} \right|^2 \\ &\leqslant K^2 \|f_0\|^2 \Delta_n \quad \text{(by Lemma E)} \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned}$$

4. A subalgebra $\mathcal{Q}$ of $B(H)$ is called transitive if it is weakly closed, contains the identity operator and has no nontrivial invariant subspaces, i.e. there exists no closed subspace $M \neq \{0\}$ and H such that $AM \subset M$ for all $A \in \mathcal{Q}$. $B(H)$ is obviously a transitive algebra. The question whether there exists a transitive algebra other than $B(H)$ is an open problem, the so-called transitive algebra problem raised by Kadison [8]. The first solution of the problem was given by Arveson [1]; see also [13]. An elegant account of such solutions is given in Radjavi and Rosenthal [15]. For later additions we refer to Lambert [9] and Yadav and Chatterjee [17]. Here we shall show that if A is a contraction of the class $C_{1\cdot}$, then the transitive algebra $\mathcal{Q}$ containing $T_{A,f}$ is $B(H)$.

THEOREM 5. *If A is a contraction belonging to $C_{1\cdot}$, then the only transitive algebra $\mathcal{Q}$ containing $T_{A,f}$ is $B(H)$.*

PROOF. Since $A \in C_{1\cdot}$ and

$$\inf_{n>0} \|A^n f\| = \mu(f) > 0 \quad \text{for } f \neq 0,$$

we have, for all n,

$$w_1 \cdots w_n \in [\mu(f)/\|f\|, 1].$$

Thus $T_{A,f}$ is similar to the multiplicity 1 unilateral shift U. Arveson [1] showed that any transitive algebra containing U is $B(H)$. As similarity preserves $B(H)$ and transitivity, the only transitive algebra $\mathcal{Q}$ containing $T_{A,f}$ is $B(H)$.

Finally we state the following theorem without proof:

THEOREM 6. *If A is an invertible contraction with A^{-1} power-bounded, then the only transitive algebra $\mathcal{Q}$ containing $T_{A,f}$ is $B(H)$.*

The authors thank the referee for providing the present short and elegant proof of Theorem 5. The original proof of the theorem was based on the techniques of [13] and ran through several pages.

References

1. W. B. Arveson, *A density theorem for operator algebras*, Duke Math. J. **34** (1967), 635–647.

2. James A. Deddens, Ralph Gellar and Domingo A. Herrero, *Commutants and cyclic vectors*, Proc. Amer. Math. Soc. **43** (1974), 169–170.

3. R. G. Douglas, H. S. Shapiro and A. L. Shields, *On cyclic vectors of the backward shift*, Bull. Amer. Math. Soc. **73** (1967), 156–159.

4. ______, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble) **20** (1970), fasc. 1, 37–76.

5. Mary R. Embry, *Strictly cyclic operator algebras on a Banach space*, Pacific J. Math. **45** (1973), 443–452.

6. R. Gellar, *Cyclic vectors and parts of the spectrum of a weighted shift*, Trans. Amer. Math. Soc. **146** (1969), 69–85.

7. D. A. Herrero, *Eigenvectors and cyclic vectors for bilateral weighted shifts*, Rev. Un. Mat. Argentina **26** (1972), 26–41.

8. R. V. Kadison, *On the orthogonalization of operator representations*, Amer. J. Math. **78** (1955), 600–621.

9. Alan Lambert, *Strictly cyclic operator algebras*, Pacific J. Math. **39** (1971), 717–726.

10. ______, *Subnormality and weighted shifts*, J. London Math. Soc. (2) **14** (1976), 476–480.

11. N. K. Nikolskiĭ, *The invariant subspaces of certain completely continuous operators*, Vestnik Leningrad Univ. (7) **20** (1965), 68–77. (Russian)

12. ______, *Invariant subspaces of weighted shift operators*, Math. USSR-Sb. 3 (1967), 159–176.

13. E. A. Nordgren, H. Radjavi and P. Rosenthal, *On density of transitive algebras*, Acta Sci. Math. (Szeged) **30** (1969), 175–179.

14. M. Rabindranathan, *On cyclic vectors of weighted shifts*, Proc. Amer. Math. Soc. **44** (1974), 293–299.

15. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

16. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó, Budapest, 1970.

17. B. S. Yadav and S. Chatterjee, *On a partial solution of the transitive algebra problem*, Acta Sci. Math. (Szeged) (to appear).

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