THE CROSSED PRODUCT OF A C*-ALGEBRA
BY AN ENDOMORPHISM

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Abstract. Let $A$ be a unital, strongly amenable C*-algebra, $\sigma: A \to pAp$ a $*$-isomorphism (where $p$ is a proper projection of $A$), and $S$ an isometry such that $SxS^* = \sigma(x)$ for all $x$ in $A$. If $A$ has no nontrivial $\sigma$-invariant ideals, then $C^*(A, S)$ is simple. Furthermore, $C^*(A, S)$ is isomorphic to a corner of the crossed product of $A \otimes$ (compacts) by an automorphism.

J. Cuntz showed in [5] that the C*-algebra generated by any countable collection of isometries on Hilbert space with range projections summing to 1 is simple. An important step in his proof was the observation that this C*-algebra is generated by an AF algebra $A$ together with a single isometry $S$ normalizing $A$ (in the sense that $SAS^*$ and $S^*AS$ are both contained in $A$). Actually, the simplicity of $C^*(A, S)$ does not depend very much on the special circumstances of [5], and in fact follows from fairly mild assumptions on $A$ and $S$. Our theorem to this effect may be regarded as a generalization of Cuntz's result.

Theorem 1. Let $A$ be a strongly amenable unital C*-algebra acting on a Hilbert space $H$. Suppose that $S$ is a nonunitary isometry (i.e. $S^*S = 1 \neq SS^*$) in $L(H)$ such that

(i) $SAS^*$ and $S^*AS$ are both contained in $A$; and

(ii) the only proper (two-sided) ideal $J$ of $A$ for which $SJS^* \subseteq J$ is the zero ideal.

Then $C^*(A, S)$ is simple.

Our procedure for proving this is quite similar in outline to that followed in [5]: construct a well-behaved projection of norm one of $C^*(A, S)$ onto $A$, do the same for the enveloping C*-algebra $B$ of the *-algebra generated by $A$ and $S$, let the circle group act on $B$ by fixing $A$ and multiplying $S$ by scalars of modulus 1, and then exploit this action to show that the natural map from $B$ to $C^*(A, S)$ is an isomorphism. What is new here is how the desired norm-one projection of $C^*(A, S)$ onto $A$ is shown to exist. This is taken care of in Lemmas 2 and 3 below. After that, the argument proceeds essentially as in [5].

Let $A$ and $S$ be as in Theorem 1. We set $p = SS^*$, so $p$ is a projection in $A$ strictly less than 1. The map $\sigma: A \to A$ defined by $\sigma(x) = SxS^*$ is a *-isomorphism of $A$ with $pAp$, with left inverse $\sigma^*$ given by $\sigma^*(x) = S^*xS$. For $k > 1$, we let
\( p_k = (S^k)(S^*)^k = a^k(1), \) so \( \{ p_k \} \) is a decreasing sequence of projections in \( A. \) Our notation for the natural left and right actions of a \( C^* \)-algebra on its conjugate space is: \( (a \cdot g)(b) = g(ab), \) \( (g \cdot a)(b) = g(ab). \)

**Lemma 2.** There is a state \( f_0 \) on \( C^*(A, S) \) whose restriction to \( A \) is faithful and which satisfies \( f_0(AS^k) = 0 \) for all \( k > 1. \)

**Proof.** Our assumption that \( A \) is strongly amenable implies that for any state \( g \) on a \( C^* \)-algebra \( B \) containing \( A, \) there is a state \( f \) in the \( w^* \)-closed convex hull of \( \{ u \cdot g \cdot u^* : u \) unitary in \( A \} \) which is centralized by \( A \) (i.e. \( f \cdot x = x \cdot f \) for \( x \) in \( A) \) [2]. In particular, \( A \) has a tracial state. No tracial state of \( A \) can vanish at the projection \( p; \) the left (= right) kernel of such a state would be a proper ideal containing \( pAp (= SAS^*), \) contrary to (ii). This permits us to consider the map of the set of tracial states of \( A \) into itself which sends a tracial state \( \tau \) to the tracial state \( \tau(p)^{-1}(\tau \circ \sigma). \) The Schauder fixed point theorem [7] gives us a tracial state \( \tau_0 \) of \( A \) and \( 0 < r < 1 \) such that \( \tau_0 \circ \sigma = r \tau_0. \) It follows from (ii) that \( \tau_0 \) is faithful and hence, because \( 1 - p \neq 0, \) we must have \( r < 1. \) Extend \( \tau_0 \) to a state \( g \) on \( C^*(A, S). \) All of the states \( u \cdot g \cdot u^* \) \( (u \) unitary in \( A) \) also extend \( \tau_0. \) Using the strong amenability of \( A, \) we thus obtain a state extension \( f \) of \( \tau_0 \) to \( C^*(A, S) \) which centralizes \( A. \) Let \( K \) denote the (convex, \( w^* \)-compact) set of all such \( A \)-centralizing state extensions of \( \tau_0. \) If \( f \) belongs to \( K, \) then so does \( r^{-1}(S^* \cdot f \cdot S) \), because

\[
(S^* \cdot f \cdot S)(xY) = f(SxS^*SYS^*) = f(\sigma(x)SYS^*)
\]

\[
= f(SYS^*\sigma(x)) = (S^* \cdot f \cdot S)(Yx) \quad (Y \in C^*(A, S)).
\]

Another application of the Schauder fixed point theorem now yields a state \( f_0 \) on \( C^*(A, S) \) extending \( \tau_0, \) centralizing \( A, \) and satisfying \( f_0 = r^{-1}(S^* \cdot f \cdot S). \) This is the state we want because for \( x \) in \( A \) and \( k > 1, \) we have

\[
f_0(xS^k) = r^{-k}f_0(S^kxS^*(S^*)^k)
\]

\[
= r^{-k}f_0(xp_kS^k) = r^{-k}f_0(xS^k).
\]

Since \( 0 < r < 1, \) this means that \( f_0(xS^k) = 0, \) and the lemma is proved.

Let \( B_0 \) be the \( * \)-algebra generated by \( A \) and \( S. \) Algebraic manipulations using (i) show that finite sums of the form

\[
\sum_{1}^{N} (S^*)^k x_{-k} + x_0 + \sum_{1}^{N} x_k S^k \quad (x's \ in \ A)
\]

constitute a \( * \)-algebra. (For instance, with \( k > j > 0, \) we have \( (S^*)^kx_{-k}x_jS^j = (S^*)^k \cdot [(S^*)^jx_{-k}x_jS^j], \) where the factor in brackets belongs to \( A.) \) Hence every operator in \( B_0 \) can be written in the form (\( * \)).

**Lemma 3.** (a) There is a projection \( E: C^*(A, S) \to A \) of norm one satisfying \( E(AS^k) = 0 \) = \( E((S^*)^kA) \) for \( k > 1. \)

(b) If \( X \) in \( B_0 \) is written in the form (\( * \)), the coefficients \( x_j \) are uniquely determined by \( X \) if we require that \( x_k p_k = x_k \) and \( p_k x_{-k} = x_{-k} \) \( (k > 1). \)
(c) If $\pi: A \to L(\mathcal{H})$ is a $*$-representation and $\widetilde{S}$ in $L(\mathcal{H})$ is an isometry such that $\widetilde{S}\pi(x)\widetilde{S}^* = \pi(\sigma(x))$, then $B_0$ and the $*$-algebra $\widetilde{B}_0$ generated by $\pi(A)$ and $\widetilde{S}$ are $*$-isomorphic (with the isomorphism sending $x \to \pi(x)$ and $S \to \widetilde{S}$).

**Proof.** (a) It will suffice to show that if $X$ in $B_0$ is written in the form (*), then $\|X\| > \|x_0\|$. For this, consider the GNS representation $(\pi_0, \xi_0, H_0)$ of $C^*(A, S)$ associated with the state $f_0$ of Lemma 1. Let $H_x = \pi_0(A)\xi_0$. The norm of the compression of $\pi_0(X)$ to $H_x$ is

$$\sup\{|f_0(w^*Xy)|: w, y \in A, f_0(w^*w) < 1, f_0(y^*y) < 1\}.$$

The “zero term” of $w^*Xy$ is $w^*x_0y$ (since $w^*(S^*)^k = (S^*)^k\sigma^k(w^*)$ and $S^*y = \sigma^k(y)S^k$), so $f_0(w^*Xy) = f_0(w^*x_0y)$. The supremum above is therefore just the norm of the image of $x_0$ under the GNS representation of $A$ arising from $f_0$. This is $\|x_0\|$, since $f_0$ is faithful, and so we have $\|X\| > \|\pi_0(X)\| > \|x_0\|$, proving part (a).

(b) This follows immediately from the observation that $E(X(S^*)^k) = x_0^k p_k$ and $E(S^kX) = p_k x_0^k (k > 1)$.

(c) Notice that the kernel of $\pi$ is a $\sigma$-invariant ideal, so $\pi$ is faithful by (ii). We have $\tilde{S}\tilde{S}^* = \pi(p)$, and thus $\tilde{S}\pi(x)\tilde{S}^* = \tilde{S}\pi(p)(x)\tilde{S}^* = \tilde{S}\pi(\sigma^{-1}(p))(x)\tilde{S}^* = \pi(\sigma(x))^\pi(x)$ for $x$ in $A$. Everything we have proved so far about $A$ and $S$ is true also of $\pi(A)$ and $\tilde{S}$. In particular, application of (b) above to $B_0$ and $\tilde{B}_0$ shows that there is a bijective linear map $\alpha: B_0 \to \tilde{B}_0$ such that $\alpha((S^*)^kx) = (S^*)^\pi(x)$ and $\alpha(xS^k) = \pi(\sigma(x))^\pi(x)(S^k)$ for $k > 0$ and $x$ in $A$. This is obviously a $*$-map, and a direct computation shows that it is multiplicative.

**Proof of Theorem 1.** Let $B$ be the completion of $B_0$ in its greatest $C^*$-norm. (This makes sense because for any $X$ in $B_0$ and any $*$-representation $\pi$ of $B_0$ on Hilbert space, $\|\pi(X)\|$ does not exceed the sum of the norms of the coefficients $x_j$ in (*).) We have a $*$-monomorphism $\theta: A \to B$, an isometry $T$ in $B$ (normalizing $\theta(A)$ in the same way that $S$ normalizes $A$) such that $B = C^*(\theta(A), T)$, and, because of the universal property of $B$, a $*$-homomorphism $\pi: B \to C^*(A, S)$ such that $\pi \circ \theta = \text{id}_A$ and $\pi(T) = S$. Using part (c) of Lemma 3 and, again, the universal property of $B$, we obtain a homomorphism $\rho$ from the circle group into the group of $*$-automorphisms of $B$ such that $\rho_x(T) = \lambda T$, $\rho_x(\theta(x)) = \theta(x)$ ($|\lambda| = 1, x$ in $A$). Checking first on $B_0$ shows that the map $\lambda \to \rho_\lambda(Y)$ is norm-continuous for each $Y$ in $B$. We can therefore define a norm-one projection $F: B \to \theta(A)$ by

$$F(Y) = \int_{|\lambda|=1} \rho_\lambda(Y) \, d\lambda.$$

(The reason that the range of $F$ is precisely $\theta(A)$ is that $F(B_0) = \theta(A)$.) This projection is faithful ($F(Y^*Y) = 0$ implies $Y = 0$) and one checks readily that $\pi \circ F = E \circ \pi$, where $E: C^*(A, S) \to A$ is as in Lemma 3. It is now easy to show that $\pi$ is an isomorphism. Indeed, we have $\pi(F(\ker \pi)) = E(\pi(\ker \pi)) = (0)$, and since $\pi$ is injective on $F(B) = \theta(A)$, this means that $F(\ker \pi) = (0)$. But $F$ is faithful, so $\ker \pi = 0$. Suppose finally that $\pi'$ is an arbitrary $*$-representation of $B$. By (ii), the restriction of $\pi'$ to $\theta(A)$ is faithful, so $\pi'(B)$ is generated by an isomorphic copy of $A$ and a normalizing isometry $\pi(S)$. Using the norm-one
projection \( E' : \pi'(B) \to \pi'(\theta(A)) \) that one obtains from Lemma 3 in place of \( E \) in the argument above, we deduce that \( \pi' \) must be faithful. Hence \( C^*(A, S) \), which is isomorphic to \( B \), is simple.

The algebra \( C^*(A, S) \) is always amenable [9], hence nuclear [4] (see also [3]), but does not admit a tracial state and therefore cannot be strongly amenable. One can think of \( C^*(A, S) \) as the crossed product of \( A \) by the endomorphism \( \sigma \). Indeed, given any \("\sigma\)-covariant" pair \((\pi, \tilde{S})\) as in Lemma 3(c), that lemma and the proof of Theorem 1 ensure that \( C^*(A, S) \) and \( C^*(\pi(A), \tilde{S}) \) are isomorphic in a natural way. We remark that if \( A \) is any unital \( C^* \)-algebra, \( p \) a nonzero projection in \( A \), and \( \sigma : A \to pAp \) a *-isomorphism (should one exist), then there exists a \( \sigma \)-covariant representation of \( A \). An easy way to see this is to define \( \sigma^* : A \to A \) by \( \sigma^*(x) = \sigma^{-1}(pxp) \), note that composition with \( \sigma^* \) takes states of \( A \) to states, and use the Schauder fixed point theorem to obtain a state \( f \) of \( A \) such that \( f \circ \sigma^* = f \) (and hence \( f \circ \sigma = f \)). If \((\pi, \xi, H)\) is the GNS representation of \( A \) associated with \( f \), one checks readily that the equation \( \tilde{S}\pi(x) = \sigma(x)\xi \) defines an isometry \( \tilde{S} \) satisfying \( \tilde{S}\pi(x)\tilde{S}^* = \pi(\sigma(x)) \) for \( x \) in \( A \).

It turns out that under the hypotheses of Theorem 1, \( C^*(A, S) \) can be obtained from a crossed product of the standard sort. An explicit construction is given in [5] to show this for the situation considered there; our result below is of necessity somewhat less detailed. Here, \( \mathcal{K} \) denotes the \( C^* \)-algebra of compact operators on a separable infinite dimensional Hilbert space.

**Theorem 2.** Let \( A \) and \( S \) be as in Theorem 1. There exist a *-automorphism \( \theta \) of \( A \otimes \mathcal{K} \) and a projection \( Q \) in the crossed product \( C^*(A \otimes \mathcal{K}, \theta) \) such that \( C^*(A, S) \) is isomorphic to \( QC^*(A \otimes \mathcal{K}, \theta)Q \).

**Proof.** Let \( B = C^*(A, S) \). From the proof of Theorem 1, we know that there is a continuous action \( \rho \) of the circle group on \( B \) such that \( \rho_x(x) = x \) and \( \rho_x(S) = \lambda S \) (\( |\lambda| = 1 \), \( x \) in \( A \)). Recall that the crossed product \( C^*(B, \rho) \) of \( B \) by the action \( \rho \) is the completion in the greatest \( C^* \)-norm of \( C(T, B) \), the space of continuous \( B \)-valued functions on the circle group, considered as a *-algebra with multiplication and involution defined by

\[ (FG)(\lambda) = \int_{|\mu|=1} F(\mu)\rho_{\mu}(G(\bar{\mu}\lambda)) \, d\mu \quad (F^*)(\lambda) = \rho_\lambda(F(\bar{\lambda})^*). \]

Let \( P \) in \( C(T, B) \) be the function with constant value 1. It is immediate that \( P \) is a projection, and, using the fact that \( A \) is the fixed-point algebra of \((B, \rho)\), one checks (see [10]) that \( A \) and \( PC^*(B, \rho)P \) are isomorphic via the map that sends \( x \) in \( A \) to the function in \( C(T, B) \) with constant value \( x \). We now claim that the closed two-sided ideal \( L \) of \( C^*(B, \rho) \) generated by \( P \) is all of \( C^*(B, \rho) \). For this, consider products of the form \( FPG \), where \( F \) is the function constantly equal to some given \( X \) in \( B \), and \( G(\lambda) = \rho_y(Y) \) for some \( Y \) in \( B \). The convolution formula gives \( (FPG)(\lambda) = X\rho_\lambda(Y) \). In particular, if \( X = (S^*)^k \) and \( Y = yS^j \) (\( k, j > 0 \), \( y \) in \( A \)), then if \( j > k \), we have \( (FPG)(\lambda) = \lambda^j(\sigma^*)^k(y)S^{j-k} \), while if \( j < k \), we obtain \( (FPG)(\lambda) = \lambda^j(S^*)^{k-j}(\sigma^*)(y) \). Since \( \sigma^* \) is surjective, it follows that all functions of
the form
\[ \lambda \rightarrow \lambda^{n}(S^{*})^{n}x \quad \text{or} \quad \lambda \rightarrow \lambda^{m}xS^{m} \quad (n > 0, m > 0, x \in A) \]
belong to \( L \). To show that \( m < 0 \) is also allowed here, take \( X = xS^{k} \) and \( Y = (S^{*})^{l}y \) and suppose that \( j > k \). We have
\[
(FPG)(\lambda) = X \rho_{\lambda}(Y) = \lambda^{-j}xp_{k}(S^{*})^{l-k}y \\
= \lambda^{-j}(S^{*})^{l-k}x^{j-k}(x)p_{k}y.
\]
Now as \( x \) ranges over \( A, (S^{*})^{l-k}x^{j-k}(x)p_{j} \) ranges over \( (S^{*})^{l-k}p_{j-k}A_{j-k}p_{j} \) (= \( (S^{*})^{l-k}Ap_{j} \)). By part (ii) of the assumption on \( A \) and \( S \), the set \( \{w, v \in A\} \) spans \( A \). (This is because \( \rho(p_{j}) = p_{j+1} \leq p_{j} \).) For \( n > 0, m < 0 \), and any \( x \in A \), the function \( \lambda \rightarrow \lambda^{m}(S^{*})^{n}x \) belongs to \( L \). A similar argument with \( j < k \) shows that the same is true of the function \( \lambda \rightarrow \lambda^{m}xS^{n} \). It now follows that for any continuous \( f: T \rightarrow \mathbb{C} \), the functions
\[ \lambda \rightarrow f(\lambda)(S^{*})^{n}x \quad \text{and} \quad \lambda \rightarrow f(\lambda)xS^{n} \]
belong to \( L \). A straightforward partition-of-unity argument shows that these functions span a sup-norm dense subspace of \( C(T, B) \). But the sup-norm on \( C(T, B) \) dominates the \( L^{1} \)-norm, which in turn dominates the greatest \( C^{*} \)-norm, so \( L = C^{*}(B, \rho) \) as claimed. In other words, the “corner” \( PC^{*}(B, \rho)P \) of \( C^{*}(B, \rho) \) to which \( A \) is isomorphic is a full corner. By Corollary 2.6 of [1], then, \( A \otimes \mathbb{K} \) and \( C^{*}(B, \rho) \otimes \mathbb{K} \) are isomorphic. (Application of this result requires that \( C^{*}(B, \rho) \) have a strictly positive element. This is not a problem here. Any scalar-valued function in \( C(T, B) \) all of whose Fourier coefficients are positive is a strictly positive element of \( C^{*}(B, \rho) \).) Let \( \hat{\rho} \) be the \( \ast \)-automorphism that generates the action dual to \( \rho \) on \( C^{*}(B, \rho) \) (see [11]), and let \( \theta \) be the automorphism of \( A \otimes \mathbb{K} \) corresponding to \( \hat{\rho} \otimes \text{id}_{\mathbb{K}} \). By H. Takai’s duality theorem [11], the crossed product \( C^{*}(A \otimes \mathbb{K}, \theta) \) is isomorphic to \( B \otimes \mathbb{K} \otimes \mathbb{K} \), so \( B \) is isomorphic to \( QC^{*}(A \otimes \mathbb{K}, \theta)Q \) for an appropriate projection \( Q \) in \( C^{*}(A \otimes \mathbb{K}, \theta) \).

We conclude with a remark about KMS states for what may be termed the “natural” dynamics on \( C^{*}(A, S) \). Let \( A, S, \sigma, E, \) and \( \rho \) be as in Theorem 1 and its proof. (Regard \( \rho \) as a 2\( \pi \)-periodic action of \( \mathbb{R} \) on \( C^{*}(A, S) \): \( \rho_{\rho}(x) = x \) for \( x \) in \( A, \rho_{\rho}(S) = \exp(itS) \).) An appropriate modification of the argument in [8] shows that for \( 0 < \beta < \infty \), the \( \beta \)-KMS states (if there are any) of the \( C^{*} \)-dynamical system \( (C^{*}(A, S), \rho) \) are precisely those of the form \( \tau \circ E \), where \( \tau \) is a tracial state on \( A \) such that \( \tau \circ \sigma = \exp(-\beta)\tau \). (See also the last paragraph of [6].) In light of this, it would appear that the algebras we have investigated here could serve as a useful source of examples of \( C^{*} \)-dynamical systems exhibiting various sorts of unusual KMS-phenomena.

**References**


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