SIMPLE PROOF OF A THEOREM OF ERDÖS AND LAX

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Abstract. An elementary new and simple proof of Erdös-Lax theorem is given which in essence involves no analysis.

Let $P(z)$ be a polynomial of degree $n$ with $\max_{|z|=1}|P(z)| = 1$, then

$$\max_{|z|=1}|P'(z)| < n. \quad (1)$$

Inequality (1) is an immediate consequence of S. Bernstein’s theorem on the derivative of a trigonometric polynomial (for reference see [5]). Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, P. Erdös conjectured and later P. D. Lax [3] proved the following

**Theorem A.** If $P(z)$ is a polynomial of degree $n$ with $\max_{|z|=1}|P(z)| = 1$ and $P(z)$ has no zeros in the disk $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| < \frac{n}{2}. \quad (2)$$

The result is best possible and equality in (2) holds for $P(z) = (\alpha + \beta z^n)/2$, where $|\alpha| = |\beta| = 1$.

For other proofs of Theorem A see [1], [2], and [4]. In this paper we give an apparently new proof of Theorem A, which in essence involves no analysis. The proof depends on the following lemma which is also of independent interest.

**Lemma 1.** If $P(z)$ is a polynomial of degree $n$ and $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + a$, where $a \neq -1$ is any nonzero complex number, then for any complex number $t$,

$$tP'(t) = \frac{n}{1 + a} P(t) + \frac{1 + a}{na} \sum_{k=1}^{n} P(tz_k) \frac{z_k}{(z_k - 1)^2}. \quad (3)$$

**Proof of Lemma 1.** Let $t$ be an arbitrary complex number. Consider the function $F_i(z)$ defined by $F_i(z) = (P(tz) - P(t))/(z - 1)$. Then $F_i(z)$ is a polynomial of degree $\leq n - 1$, and therefore, by using Lagrange’s interpolation formula with $z_1, z_2, \ldots, z_n$ as the basic points of interpolation we can write $F_i(z)$ as

$$F_i(z) = \sum_{k=1}^{n} F_i(z_k) \frac{z^n + a}{nz_k^n - (z - z_k)}. \quad (3)$$
Since \( z_k^n + a = 0 \) and \( z_k \neq 0, k = 1, 2, \ldots, n \), therefore, \( z_k^{n-1} = -a/z_k \) and we get
\[
F_k(z) = \frac{1}{na} \sum_{k=1}^{n} F_k(z_k) \frac{z_k(z^n + a)}{(z_k - z)}.
\]

Now using the fact \( F_k(1) = tP'(t) \), we obtain the following identity in \( t \).
\[
tP'(t) = \frac{1}{na} \sum_{k=1}^{n} F_k(z_k) \frac{z_k(1 + a)}{(z_k - 1)} = \frac{1 + a}{na} \sum_{k=1}^{n} (P(tz_k) - P(t)) \frac{z_k}{(z_k - 1)^2} \\
= \frac{1 + a}{na} \sum_{k=1}^{n} P(tz_k) \frac{z_k}{(z_k - 1)^2} - \frac{(1 + a)P(t)}{na} \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2}. \tag{4}
\]

Setting \( P(t) = t^n \) in (4) we get
\[
(1/n) \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2} = -\frac{na}{(1 + a)^2}. \tag{5}
\]

Combining (4) and (5) we get (3) and therefore the lemma is established.

From Lemma 1, we now deduce the following

**Lemma 2.** If \( P(z) \) is a polynomial of degree \( n \), then
\[
|P'(z)| + |nP(z) - zP'(z)| \leq n \max_{|z|=1} |P(z)| \quad \text{for} \quad |z| = 1. \tag{6}
\]

**Proof of Lemma 2.** In Lemma 1 we take \( a \neq -1 \) to be an arbitrary complex number such that \( |a| = 1 \), then the zeros \( z_k \) of \( z^n + a \) are of unit modulus and \( z_k \neq 1, k = 1, 2, \ldots, n \). So that from (3) for \( |t| = 1 \) we obtain
\[
|atP'(t) + tP'(t) - nP(t)| = \left| \frac{(1 + a)^2}{na} \sum_{k=1}^{n} P(tz_k)z_k/(z_k - 1)^2 \right| \\
\leq \left| \frac{(1 + a)^2}{na} \right| \sum_{k=1}^{n} \left| z_k/(z_k - 1)^2 \right| \max_{|z|=1} |P(t)|. \tag{7}
\]

Now if \( |z| = 1 \) and \( z \neq 1 \) then \( z/(z - 1)^2 \) is a negative real number. In fact it can be easily seen that
\[
eq -1/4 \sin^2 \theta/2, \theta \neq O(2\pi) \quad \text{and moreover for} \quad |a| = 1 \quad \text{and} \quad a \neq -1, \quad (1 + a^2)/a \quad \text{is a positive real number. Therefore,}
\]
\[
\left| \frac{(1 + a)^2}{na} \right| \sum_{k=1}^{n} \left| z_k/(z_k - 1)^2 \right| = \frac{(1 + a)^2}{na} \sum_{k=1}^{n} z_k/(z_k - 1)^2 \\
= n \quad \text{(by (5)).}
\]

Hence from (7) it follows that
\[
|atP'(t) + tP'(t) - nP(t)| \leq n \max_{|t|=1} |P(t)| \quad \text{for} \quad |t| = 1, |a| = 1, a \neq -1.
\]

This equality obviously holds for \( a = -1 \) also. Choosing the argument of \( a \) such that
\[
|atP'(t) + tP'(t) - nP(t)| = |P'(t)| + |nP(t) - tP'(t)|,
\]
we get

$$|P'(z)| + |nP(z) - zP'(z)| \leq n \max_{|z|=1} |P(z)|$$

for $|z| = 1$.

This is equivalent to the desired result.

**Proof of Erdös-Lax Theorem.** Since the polynomial $P(z)$ does not vanish in the disk $|z| < 1$, we can write $P(z) = c\prod_{j=1}^{n} (z - w_j)$ where $|w_j| > 1, j = 1, 2, \ldots, n$. Now for points $e^{i\theta}, 0 < \theta < 2\pi$ other than the zeros of $P(z)$ we have

$$\text{Re}(e^{i\theta}P'(e^{i\theta}))/nP(e^{i\theta}) = \frac{1}{n} \sum_{j=1}^{n} \text{Re}(e^{i\theta}/(e^{i\theta} - w_j))$$

$$< (1/n) \sum_{j=1}^{n} (1/2) = 1/2.$$ 

This implies

$$|e^{i\theta}P'(e^{i\theta})/nP(e^{i\theta})| < |1 - (e^{i\theta}P'(e^{i\theta}))/nP(e^{i\theta})|$$

for points $e^{i\theta}$ other than the zeros of $P(z)$. Equivalently

$$|P'(e^{i\theta})| < |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|$$

(8)

for points $e^{i\theta}$ other than the zeros of $P(z)$. Since the inequality (8) is trivially true for points $e^{i\theta}$ which are the zeros of $P(z)$, therefore, it follows that

$$|P'(z)| < |nP(z) - zP'(z)|$$

for $|z| = 1$. (9)

Combining the inequality (9) with the conclusion of Lemma 2, we get

$$|P'(z)| < (n/2) \max_{|z|=1} |P(z)|$$

for $|z| = 1$.

This is equivalent to the desired result.

If $P(z)$ is a polynomial of degree $n$, then obviously

$$|nP(z) - zP'(z)| + |zP'(z)| > n|P(z)|.$$ 

This gives

$$\max_{|z|=1} (|nP(z) - zP'(z)| + |P'(z)|) > n \max_{|z|=1} |P(z)|.$$ 

With the help of this inequality we can restate Lemma 2 as follows.

**Lemma 3.** If $P(z)$ is a polynomial of degree $n$, then

$$\max_{|z|=1} (|P'(z)| + |nP(z) - zP'(z)|) = n \max_{|z|=1} |P(z)|.$$ 

(10)

**Remark 1.** If $P(z)$ is a self-inversive polynomial, i.e. if $P(z) = Q(z)$, where $Q(z) = z^nP(1/z)$, then $|Q'(z)| = |nP(z) - zP'(z)|$ for $|z| = 1$ and it easily follows from Lemma 3 that

$$\max_{|z|=1} |P'(z)| = (n/2) \max_{|z|=1} |P(z)|.$$ 

Many other interesting results follow easily from Lemma 1. For example the following theorem is an immediate consequence of Lemma 1.
Theorem. If \( P(z) \) is a polynomial of degree \( n \) such that \( P(\beta) = 0 \), then
\[
|\beta P'(\beta)| \leq \left(\frac{n}{2}\right) \max_{1 \leq k \leq n} |P(\beta z_k)|
\]
where \( z_k, \ k = 1, 2, \ldots, n, \) are the zeros of \( z^n + 1 \). The result is sharp.

Corollary. If \( P(z) \) is a polynomial of degree \( n \) such that \( P(1) = 0 \), then
\[
|P'(1)| \leq \left(\frac{n}{2}\right) \max_{1 \leq k \leq n} |P(z_k)|
\]
where \( z_k, \ k = 1, 2, \ldots, n, \) are the zeros of \( z^n + 1 \). The result is best possible and equality holds for \( P(z) = z^n - 1 \).

References

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