SIMPLE PROOF OF A THEOREM OF ERDÖS AND LAX

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Abstract. An elementary new and simple proof of Erdös-Lax theorem is given which in essence involves no analysis.

Let $P(z)$ be a polynomial of degree $n$ with $\text{Max}_{|z|=1}|P(z)| = 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \leq n. \quad (1)$$

Inequality (1) is an immediate consequence of S. Bernstein’s theorem on the derivative of a trigonometric polynomial (for reference see [5]). Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, P. Erdös conjectured and later P. D. Lax [3] proved the following

**Theorem A.** If $P(z)$ is a polynomial of degree $n$ with $\text{Max}_{|z|=1}|P(z)| = 1$ and $P(z)$ has no zeros in the disk $|z| < 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \leq \frac{n}{2}. \quad (2)$$

The result is best possible and equality in (2) holds for $P(z) = (\alpha + \beta z^n)/2$, where $|\alpha| = |\beta| = 1$.

For other proofs of Theorem A see [1], [2], and [4]. In this paper we give an apparently new proof of Theorem A, which in essence involves no analysis. The proof depends on the following lemma which is also of independent interest.

**Lemma 1.** If $P(z)$ is a polynomial of degree $n$ and $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + a$, where $a \neq -1$ is any nonzero complex number, then for any complex number $t$,

$$tP'(t) = \frac{n}{1 + a} P(t) + \frac{1 + a}{na} \sum_{k=1}^{n} P(tz_k) \frac{z_k}{(z_k - 1)^2}. \quad (3)$$

**Proof of Lemma 1.** Let $t$ be an arbitrary complex number. Consider the function $F_i(z)$ defined by $F_i(z) = (P(tz) - P(t))/(z - 1)$. Then $F_i(z)$ is a polynomial of degree $\leq n - 1$, and therefore, by using Lagrange’s interpolation formula with $z_1, z_2, \ldots, z_n$ as the basic points of interpolation we can write $F_i(z)$ as

$$F_i(z) = \sum_{k=1}^{n} F_i(z_k) \frac{z^n + a}{n z_k^{n-1}(z - z_k)}. \quad (4)$$

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Since \( z_k^n + a = 0 \) and \( z_k \neq 0, k = 1, 2, \ldots, n \), therefore, \( z_k^{n-1} = -a/z_k \) and we get
\[
F_i(z) = (1/na) \sum_{k=1}^{n} F_i(z_k) \frac{z_k (z^n + a)}{(z_k - z)}.
\]

Now using the fact \( F_i(1) = tP'(t) \), we obtain the following identity in \( t \).
\[
tP'(t) = \frac{1}{na} \sum_{k=1}^{n} F_i(z_k) \frac{z_k (1 + a)}{(z_k - 1)} = \frac{1 + a}{na} \sum_{k=1}^{n} \left( P(tz_k) - P(t) \right) \frac{z_k}{(z_k - 1)^2}
\]
\[
= \frac{1 + a}{na} \sum_{k=1}^{n} P(tz_k) \frac{z_k}{(z_k - 1)^2} - \frac{(1 + a)P(t)}{na} \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2}.
\]  \hspace{1cm} (4)

Setting \( P(t) = t^n \) in (4) we get
\[
(1/n) \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2} = -\frac{na}{(1 + a)^2}.
\]  \hspace{1cm} (5)

Combining (4) and (5) we get (3) and therefore the lemma is established.

From Lemma 1, we now deduce the following

**Lemma 2.** If \( P(z) \) is a polynomial of degree \( n \), then
\[
|P'(z)| + |nP(z) - zP'(z)| < n \text{Max}_|z|=1 |P(z)| \text{ for } |z| = 1.
\]  \hspace{1cm} (6)

**Proof of Lemma 2.** In Lemma 1 we take \( a \neq -1 \) to be an arbitrary complex number such that \( |a| = 1 \), then the zeros \( z_k \) of \( z^n + a \) are of unit modulus and \( z_k \neq 1, k = 1, 2, \ldots, n \). So that from (3) for \( |t| = 1 \) we obtain
\[
|atP'(t) + tP'(t) - nP(t)| = \left| \frac{(1 + a)^2}{na} \sum_{k=1}^{n} P(tz_k) \frac{z_k}{(z_k - 1)^2} \right|
\]
\[
< \left| \frac{(1 + a)^2}{na} \sum_{k=1}^{n} \left| \frac{z_k}{(z_k - 1)^2} \right| \text{Max}_|t|=1 |P(t)|. \]  \hspace{1cm} (7)

Now if \( |z| = 1 \) and \( z \neq 1 \) then \( z/(z - 1)^2 \) is a negative real number. In fact it can be easily seen that \( e^\theta/(e^\theta - 1)^2 = -1/4 \sin^2\theta/2, \theta = O(2\pi) \) and moreover for \( |a| = 1 \) and \( a \neq -1 \), \((1 + a)^2/a \) is a positive real number. Therefore,
\[
\left| \frac{(1 + a)^2}{na} \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2} \right| = \frac{(1 + a)^2}{na} \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2}
\]
\[
= n \quad \text{(by (5)).}
\]

Hence from (7) it follows that
\[
|atP'(t) + tP'(t) - nP(t)| < n \text{Max}_|t|=1 |P(t)| \text{ for } |t| = 1, |a| = 1, a \neq -1.
\]

This equality obviously holds for \( a = -1 \) also. Choosing the argument of \( a \) such that
\[
|atP'(t) + tP'(t) - nP(t)| = |P'(t)| + |nP(t) - tP'(t)|,
\]
we get

\[ |P'(z)| + |nP(z) - zP'(z)| \leq n \max_{|z|=1} |P(z)| \quad \text{for } |z| = 1. \]

This is equivalent to the desired result.

**Proof of Erdős-Lax Theorem.** Since the polynomial \( P(z) \) does not vanish in the disk \( |z| < 1 \), we can write \( P(z) = c \prod_{j=1}^{n} (z - w_j) \) where \( |w_j| > 1 \), \( j = 1, 2, \ldots, n \). Now for points \( e^{i\theta}, 0 < \theta < 2\pi \) other than the zeros of \( P(z) \) we have

\[
\text{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{nP(e^{i\theta})}\right) = \frac{1}{n} \sum_{j=1}^{n} \text{Re}\left(\frac{e^{i\theta}/(e^{i\theta} - w_j)}{P(e^{i\theta})}\right)
\]

\[
< \frac{1}{n} \sum_{j=1}^{n} (1/2) = 1/2.
\]

This implies

\[
|e^{i\theta}P'(e^{i\theta})/nP(e^{i\theta})| < |1 - (e^{i\theta}P'(e^{i\theta}))/nP(e^{i\theta})|
\]

for points \( e^{i\theta} \) other than the zeros of \( P(z) \). Equivalently

\[
|P'(e^{i\theta})| < |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|
\]

for points \( e^{i\theta} \) other than the zeros of \( P(z) \). Since the inequality (8) is trivially true for points \( e^{i\theta} \) which are the zeros of \( P(z) \), therefore, it follows that

\[
|P'(z)| < |nP(z) - zP'(z)| \quad \text{for } |z| = 1.
\]

Combining the inequality (9) with the conclusion of Lemma 2, we get

\[
|P'(z)| < \left(\frac{n}{2}\right) \max_{|z|=1} |P(z)| \quad \text{for } |z| = 1.
\]

This is equivalent to the desired result.

If \( P(z) \) is a polynomial of degree \( n \), then obviously

\[
|nP(z) - zP'(z)| + |zP'(z)| > n |P(z)|.
\]

This gives

\[
\max_{|z|=1} (|nP(z) - zP'(z)| + |P'(z)|) > n \max_{|z|=1} |P(z)|.
\]

With the help of this inequality we can restate Lemma 2 as follows.

**Lemma 3.** If \( P(z) \) is a polynomial of degree \( n \), then

\[
\max_{|z|=1} (|P'(z)| + |nP(z) - zP'(z)|) = n \max_{|z|=1} |P(z)|.
\]

**Remark 1.** If \( P(z) \) is a self-inversive polynomial, i.e. if \( P(z) = Q(z) \), where \( Q(z) = z^n P(1/z) \), then \( |Q'(z)| = |nP(z) - zP'(z)| \) for \( |z| = 1 \) and it easily follows from Lemma 3 that

\[
\max_{|z|=1} |P'(z)| = \left(\frac{n}{2}\right) \max_{|z|=1} |P(z)|.
\]

Many other interesting results follow easily from Lemma 1. For example the following theorem is an immediate consequence of Lemma 1.
Theorem. If \( P(z) \) is a polynomial of degree \( n \) such that \( P(\beta) = 0 \), then
\[
|\beta P'(\beta)| \leq \left(\frac{n}{2}\right) \max_{1 \leq k \leq n} |P(\beta z_k)|
\]
where \( z_k, k = 1, 2, \ldots, n \), are the zeros of \( z^n + 1 \). The result is sharp.

Corollary. If \( P(z) \) is a polynomial of degree \( n \) such that \( P(1) = 0 \), then
\[
|P'(1)| \leq \left(\frac{n}{2}\right) \max_{1 \leq k \leq n} |P(z_k)|
\]
where \( z_k, k = 1, 2, \ldots, n \), are the zeros of \( z^n + 1 \). The result is best possible and equality holds for \( P(z) = z^n - 1 \).

References

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