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ON POWER COMPACT OPERATORS

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ABSTRACT. We give an operator theoretic proof of the following result of D. G. Tacon:

THEOREM. *If* $\{T_n\}$ *is a sequence of bounded linear operators in a complex infinite dimensional Hilbert space with the property that for every bounded sequence* $\{x_n\}$ *there exists a positive integer k such that the sequence* $\{T_k x_n\}_{n=1}^{\infty}$ *has a convergent subsequence, then there exists k such that* T_k *is a compact operator.*

Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. We shall say that a sequence $\{T_n\}$ in $\mathcal{L}(\mathcal{H})$ has property (P) if for every bounded sequence $\{x_n\}$ in $\mathcal{H}$ there exists a positive integer k such that $\{T_k x_n\}_{n=1}^{\infty}$ has a convergent subsequence. In **[1]** D. G. Tacon, using techniques from nonstandard analysis, showed that every sequence with property (P) contains a compact operator. More generally, in **[1]** this result is proved for sequences of operators acting on a Banach space. The purpose of this paper is to give a proof of this theorem in the standard framework of operator theory for the case when the operators are acting in a complex Hilbert space. Our proof is based on a deep theorem of D. Voiculescu **[2]**. The result we need from **[2]** is stated as Theorem A below. First we prove two elementary lemmas.

LEMMA 1. *Let* $\{T_n\}$ *be a sequence in* $\mathcal{L}(\mathcal{H}_1)$ *with property* (P). *If* U: $\mathcal{H}_1 \to \mathcal{H}_2$ *is a unitary operator and* $\{K_n\}$ *is a sequence of compact operators in* $\mathcal{L}(\mathcal{H}_2)$, *then the sequence* $\{UT_nU^* + K_n\}$ *has property* (P).

PROOF. Property (P) is preserved by compact perturbations of the operators in the sequence, and obviously the sequence $\{UT_nU^*\}$ has property (P).

LEMMA 2. *Let* $A_n \in \mathcal{L}(\mathcal{H})$ *such that* $A_n \neq 0$ *for* $n = 1, 2, \ldots$. *Then there exists* x_0 *in* $\mathcal{H}$ *such that* $A_n x_0 \neq 0$ *for all n.*

PROOF. It is an easy consequence of the Baire category theorem.

We denote the ideal of compact operators in $\mathcal{L}(\mathcal{H})$ by $K(\mathcal{H})$, and the quotient map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/K(\mathcal{H})$ by π. Let $\mathcal{H}_\infty = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ be the direct sum of $\aleph_0$ copies of the Hilbert space $\mathcal{H}$.

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Theorem A [2]. *Let $T_n \in \mathcal{L}(\mathcal{H})$ $(n = 1, 2, \dots)$ and let $\mathcal{Q}$ be the separable C^*-subalgebra of $\mathcal{L}(\mathcal{H})/K(\mathcal{H})$ generated by $\pi(I)$ and the sequence $\{\pi(T_n)\}$. Let ρ be a faithful representation of $\mathcal{Q}$ on some separable Hilbert space $\mathcal{K}$. Then there exists a unitary operator $U: \mathcal{H} \to \mathcal{H} \oplus \mathcal{K}_\infty$ such that $UT_nU^* - T_n \oplus \rho(\pi(T_n)) \oplus \rho(\pi(T_n)) \oplus \dots$ is a compact operator for $n = 1, 2, \dots$.*

Theorem B [1]. *Let $\{T_n\}$ be a sequence in $\mathcal{L}(\mathcal{H})$ with property* (P). *Then there exists a positive integer k such that T_k is a compact operator.*

Proof. Suppose the conclusion of the theorem were false. Then $\pi(T_n) \neq 0$ for $n = 1, 2, \dots$. If we apply Theorem A to the sequence $\{T_n\}$, then $K_n + UT_nU^* = T_n \oplus A_n \oplus A_n \oplus \dots$ where K_n is a compact operator on $\mathcal{H} \oplus \mathcal{K}_\infty$ and $A_n = \rho(\pi(T_n)) \in \mathcal{L}(\mathcal{K})$. From Lemma 1 the sequence defined by $T_n' = T_n \oplus A_n \oplus A_n \oplus \dots$ has property (P). Since ρ is a faithful representation, we have that $A_n \neq 0$ for all n. Then from Lemma 2 there exists a vector y in $\mathcal{K}$ such that $A_n y \neq 0$ for all n.

Let $\{x_n\}$ be the bounded sequence in $\mathcal{H} \oplus \mathcal{K}_\infty$ defined by

$$x_n = \langle \overbrace{0, \dots, 0}^{n}, y, 0, 0, \dots \rangle \qquad (n = 1, 2, \dots).$$

Then

$$T_k' x_n = \langle \overbrace{0, \dots, 0}^{n}, A_k y, 0, 0, \dots \rangle$$

and $\|T_k' x_n - T_k' x_m\| = \sqrt{2}\,\|A_k y\|$ for $n \neq m$ and $k = 1, 2, \dots$. This shows that the sequence $\{T_n'\}$ does not have property (P). This is a contradiction. Therefore $\pi(T_k) = 0$ for some positive integer k and T_k is a compact operator.

Remark [1]. If we consider a sequence of the form $\{T^n\}$ with a fixed operator T, then property (P) is a necessary and sufficient condition for the operator T to have a compact power.

References

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