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A PROOF OF A CONJECTURE OF A. H. STONE

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ABSTRACT. In this paper we use the techniques of analytic topology to establish a conjecture of A. H. Stone: A perfectly normal, locally connected, connected space is multicoherent if and only if there exist four nonempty, closed and connected subsets A_0, A_1, A_2, A_3 of X such that $\bigcup_{i=0}^{3} A_i = X$ and the nerve of $\{A_0, A_1, A_2, A_3\}$ forms a closed 4-gon, i.e. A_i meets A_{i+1} and A_{i-1} and no others (the suffices being taken mod 4).

1. Introduction. Throughout this paper X will denote a locally connected, connected normal space. A *region* in X is an open connected subset of X and a *continuum* in X is a closed (not necessarily compact) connected subset of X. We say that X is *unicoherent* provided, whenever $X = H \cup K$ where H and K are continua, $H \cap K$ is connected and we say that X is *multicoherent* if X is not unicoherent.

In 1971, the author wrote A. H. Stone asking for his opinion concerning the following conjecture:

> X is unicoherent if and only if every pair of disjoint nonempty continua in X can be separated by a continuum in X.

Professor Stone replied that (*) seemed to be a special case ($n = 4$) of one of his conjectures. Let $n > 2$ be an integer. We define the conjecture $S(n)$ by:

> $S(n) \equiv X$ is multicoherent if and only if X can be represented as the union of n closed and connected continua, $\{A_0, A_1, \ldots, A_{n-1}\}$ whose nerve is a closed n-gon, i.e. A_i meets A_{i-1} and A_{i+1} and no others (the suffixes being taken mod n).

Stone announced $S(3)$ in [7] and stated (in a private communication) that he was able to establish $S(n)$ for all $n > 2$ only under every strong hypotheses, such as X as finitely multicoherent (see [3] for definition).

In [5], it was shown that (*) and $S(4)$ were equivalent and that each was true in the class of spaces having Property C. A space X has *Property* C if for every separated closed set B there exist disjoint continua L and M in X such that $B \subset (L \cup M)$ and $L \cap B \neq \varnothing \neq M \cap B$. In [4], an example of a uniformly locally connected, connected separable metric space without Property C was given.

Finally in [3], the author showed that $S(6)$ obtained whenever X was compact and in [1], Harold Bell and the author gave an example of a multicoherent,

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1-dimensional, plane Peano continuum for which $S(7)$ fails. Since $S(n+1)$ implies $S(n)$, this meant, that $S(n)$, for $n > 6$, was in general false.

The principal objective of this note is to prove that $S(4)$ obtains when we assume that X is a perfectly normal space.

THEOREM. *If X is a perfectly normal space then $S(4)$ (or equivalently $(*)$) holds for X.*

2. Definitions and preliminary results. A continuous surjection $f\colon X \to Y$ is *nonalternating* if for every $y \in Y$, if $X \setminus f^{-1}(y) = P \cup Q$ is a separation, then $f(P) \cap f(Q) = \varnothing$ [**8**, p. 127].

LEMMA 1 [**2**, LEMMA 1]. *A continuous surjection $f\colon X \to [-1, 1]$ is nonalternating if and only if for every $s \in (-1, 1)$, $X \setminus f^{-1}(s)$ has exactly two components, i.e. $f^{-1}([-1, s))$ and $f^{-1}((s, 1])$ are connected sets.*

A continuous surjection $f\colon X \to Y$ is *interior at* $y \in Y$, if whenever U is an open set in X, $U \cap f^{-1}(y) \neq \varnothing$, implies that y is interior to $f(U)$. The set of points in Y at which f is interior is denoted by $\mathcal{I}(f)$. Let $\mathcal{D}$ denote the set of dyadic rationals in $(-1, 1)$.

LEMMA 2. *Let X be perfectly normal and let A and B be nonempty disjoint closed subsets of X. Then there exists a nonalternating map $f\colon X \to [-1, 1]$ such that $-1 \in f(A)$, $1 \in f(B)$, $\mathcal{D} \subseteq \mathcal{I}(f)$ and $f(A \cup B) \cap \mathcal{I}(f) = \varnothing$. Furthermore, if A and B are connected, then $f(A) = -1$ and $f(B) = 1$.*

PROOF. The proof is identical to that of Lemma 4 of [**2**].

LEMMA 3. *Let $f\colon X \to [-1, 1]$ be a continuous surjection and let $-1 < s < 1$. Then $s \in \mathcal{I}(f)$ if and only if $\operatorname{cl} f^{-1}([-1, s)) \cap \operatorname{cl} f^{-1}((s, 1]) = f^{-1}(s)$. In particular if $s, t \in \mathcal{I}(f)$ and $-1 < s < t < 1$, then $\operatorname{cl} f^{-1}((s, t)) = f^{-1}([s, t])$.*

PROOF. The continuity of f yields that $\operatorname{cl} f^{-1}([-1, s)) \cap \operatorname{cl} f^{-1}((s, 1]) \subseteq f^{-1}(s)$. Now suppose $s \in \mathcal{I}(f)$ and $x \in f^{-1}(s)$ and let U be any connected open set containing x. Since $f(U)$ is connected and s is interior to $f(U)$, $f(U)$ meets $[-1, s)$ and $f(U)$ meets $(s, 1]$. This implies that $x \in \operatorname{cl} f^{-1}([-1, s)) \cap \operatorname{cl} f^{-1}((s, 1])$ as required. Now suppose $f^{-1}(s) \subseteq \operatorname{cl} f^{-1}([-1, s)) \cap \operatorname{cl} f^{-1}((s, 1])$ and let $x \in f^{-1}(s)$. Then if U is any connected open set containing x, there are points $u \in U \cap f^{-1}([-1, s))$ and $v \in U \cap f^{-1}((s, 1])$. Then $s \in (u, v) \subseteq f(U)$ and f is interior at s. This completes the proof.

LEMMA 4. *Let X be perfectly normal. Then every pair of nonempty disjoint continua in X can be separated by a continuum if and only if for every nonalternating function $f\colon X \to [-1, 1]$, if $s, t \in \mathcal{I}(f)$, $-1 < s < t < 1$, then $f^{-1}((s, t))$ is connected.*

PROOF. The sufficiency: Let A and B be nonempty disjoint continua in X. By Lemma 2, there exists a nonalternating map $f\colon X \to [-1, 1]$ such that $f(A) = -1$, $f(B) = 1$ and $D \subseteq \mathcal{I}(f)$. Since f is interior at $-\frac{1}{2}$ and $\frac{1}{2}$, $\operatorname{cl} f^{-1}((-\frac{1}{2}, \frac{1}{2})) = f^{-1}([\frac{1}{2}, \frac{1}{2}])$ and so by our hypothesis $C = f^{-1}[-\frac{1}{2}, \frac{1}{2}]$ is a continuum in X. Clearly C separates A and B in X as required.

PROOF OF THE NECESSITY. Let $s < t$ and let $s, t \in \mathcal{G}(f)$. By Lemma 1, $f^{-1}([-1, s))$ and $f^{-1}((t, 1])$ are connected sets and since $s, t \in \mathcal{G}(f)$, $\operatorname{cl} f^{-1}([-1, s)) = f^{-1}([-1, s])$ and $\operatorname{cl} f^{-1}((t, 1]) = f^{-1}([t, 1])$ and each of these sets is a continuum. Now, by our hypothesis some continuum C in $f^{-1}((s, t))$ separates $f^{-1}([-1, s])$ and $f^{-1}([t, 1])$ in X. Let K be the component of $f^{-1}((s, t))$ that contains C. We assert that $f(K) = (s, t)$. To see this, suppose $s \notin \operatorname{cl} f(K)$. Then K is a component of $X \setminus f^{-1}(t)$ and so $X \setminus f^{-1}(t)$ has at least three components, K, and the components meeting $f^{-1}(-1)$ and $f^{-1}(1)$ respectively. Of course this contradicts Lemma 1 and so $s \in \operatorname{cl} f(K)$. Similarly $t \in \operatorname{cl} f(K)$ and $f(K) = (s, t)$. Now, this implies $K = f^{-1}(s, t)$. For if Q is any other component of $f^{-1}((s, t))$, it follows, by the argument above, that $f(Q) = (s, t)$. This means K cannot separate $f^{-1}([-1, s])$ and $f^{-1}([t, 1])$ and this is a contradiction. Thus $f^{-1}((s, t)) = K$ is connected and this completes the proof.

3. Proof of main result.

THEOREM. *Let X be a locally connected, connected, perfectly normal space. Then $S(4)$ holds for X.*

PROOF. We will prove the equivalent statement: $(*) \equiv X$ is unicoherent if and only if every pair of nonempty disjoint continua can be separated by a continuum. By Theorem 4.7 of [9, p. 49], if X is unicoherent, every pair of nonempty disjoint continua in X can be separated by a continuum.

Now suppose that every pair of nonempty disjoint continua in X can be separated in X by a continuum, but that X is not unicoherent. Then $X = H \cup K$ where H and K are closed and connected sets and $H \cap K = A \cup B$ where A and B are disjoint, nonempty, closed sets. By Lemma 2, there exists a nonalternating map $f\colon X \to [-1, 1]$ such that $-1 \in f(A)$, $1 \in f(B)$, $\mathfrak{D} \subseteq \mathcal{G}(f)$ and $\mathcal{G}(f) \cap f(A \cup B) = \varnothing$.

Now $f^{-1}(0)$ separates $f^{-1}(-1) \cap A$ and $f^{-1}(1) \cap B$ and $f^{-1}(0) \cap (A \cup B) = \varnothing$. This implies that $f^{-1}(0)$ is not connected, say $f^{-1}(0) = C \cup D$, where C and D are nonempty disjoint closed subsets of X. Since X is perfectly normal, there exists a continuous surjection $g\colon X \to [-1, 1]$ with $g^{-1}(-1) = C$, $g^{-1}(1) = D$ and $g^{-1}(0) = f^{-1}(\{-1, 1\})$.

By Lemma 4, $f^{-1}((-1, 1)) = \bigcup_{i=1}^{\infty} f^{-1}(-1 + 2^{-i}, 1 - 2^{-i})$ is connected. But this is impossible since $f^{-1}((-1, 1)) = X \setminus g^{-1}(0) = g^{-1}([-1, 0)) \cup g^{-1}((0, 1])$ is a separated set. Hence X is unicoherent and this completes the proof.

COROLLARY. *Let X be a locally connected, connected, perfectly normal space. Then the following are equivalent:*

Property I. *If A and B are disjoint closed subsets of X, and $x, y \in X$ are such that neither A nor B separates x and y in X then $A \cup B$ does not separate x and y in X.*

Property I′ (*Phragmen-Brouwer Property*). *If neither of the disjoint closed subsets A and B of X separates X, then $A \cup B$ does not separate X.*

Property II (*Brouwer Property*). *If M is a closed, connected subset of X and C is a component of $X \setminus M$, then the boundary of C is a closed and connected set.*

Property III (*Unicoherence*). *If $X = A \cup B$, where A and B are closed and connected, then $A \cap B$ is connected.*

Property IV. *If F is a closed subset of X, and C_1, C_2 are disjoint components of $X \setminus F$ which have the same boundary, B, then B is closed and connected.*

Property V. *If A and B are disjoint closed subsets of X, $a \in A$, $b \in B$, then there exists a closed, connected subset C of $X \setminus (A \cup B)$ which separates a and b.*

Property VI. *Every pair of disjoint continua in X can be separated by a continuum in X.*

Property VII. *For every nonalternating map $f: X \to [-1, 1]$ and every pair $s, t \in \mathcal{G}(f)$, $s < t$, $f^{-1}((s, t))$ is connected.*

Property VIII. *If neither of the disjoint closed connected sets A and B of X separates X, then $A \cup B$ does not separate X.*

PROOF. The equivalence of I–V is Theorem 4.10 of [9]. III is equivalent to VI by the theorem above and VI is equivalent to VII by Lemma 4. The equivalence of VIII and VI is Theorem 2 of [3].

REMARK. Some of the equivalences above are known to hold under weaker conditions, e.g., Theorem 4.10 of [9] assumes that X is completely normal and Theorem 2 of [3] merely requires that X be normal. In [6], J. H. V. Hunt shows that I, I′ and III are equivalent without the assumption of any separation axioms.

It is unknown whether $S(5)$ or $S(6)$ obtains for noncompact spaces.

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