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CENTRALIZER NEAR-RINGS THAT ARE ENDOMORPHISM RINGS

CARLTON J. MAXSON AND KIRBY C. SMITH

ABSTRACT. For a finite ring R with identity and a finite unital R-module V the set $C(R; V) = \{f: V \to V | f(\alpha v) = \alpha f(v) \text{ for all } \alpha \in R,\ v \in V\}$ is the centralizer near-ring determined by R and V. Those rings R such that $C(R; V)$ is a ring for every R-module V are characterized. Conditions are given under which $C(R; V)$ is a semisimple ring. It is shown that if $C(R; V)$ is a semisimple ring then $C(R; V) = \mathrm{End}_R(V)$.

1. Preliminaries. Let G be a group and Γ a semigroup of endomorphisms of G. Then $C(\Gamma; G) = \{f: G \to G | f(0) = 0 \text{ and } f(\gamma a) = \gamma f(a) \text{ for all } \gamma \in \Gamma,\ a \in G\}$ is a near-ring under the operations of function addition and function composition, and is called the centralizer near-ring determined by Γ and G. Moreover, every near-ring with identity arises in this manner [**6**, p. 50]. It has been shown by Betsch [**1**] that N is a finite simple near-ring with identity if and only if there exists a finite group G and a fixed point free group of automorphisms Γ of G such that $N \cong C(\Gamma; G)$. The structure of $C(\Gamma; G)$ for various G's and Γ's has been investigated in [**2**], [**3**] and [**4**].

Throughout this paper R will denote a finite ring with 1 and V a finite unital R-module. The corresponding centralizer near-ring is $C(R; V) = \{f: V \to V | f(rv) = rf(v) \text{ for all } r \in R,\ v \in V\}$. In dealing with $C(R; V)$ we may assume, without loss of generality, that V is a faithful R-module, for we have $C(R; V) = C(\bar{R}; V)$ where V is a faithful $\bar{R}$-module, $\bar{R} = R/\mathrm{Ann}(V)$.

In [**5**] we showed that if R is a finite simple ring then $C(R; V)$ is a simple near-ring. This result is used to obtain the following generalization.

PROPOSITION. *Let R be a finite semisimple ring and let V be a finite R-module. Then $C(R; V)$ is a semisimple near-ring.*

PROOF. We have $R = S_1 \oplus \cdots \oplus S_t$ where each S_i is a simple ring. Let e_i denote the identity of S_i. If $V_i = \{v \in V | e_i v = v\}$ then $V = V_1 \oplus \cdots \oplus V_t$ and $f(V_i) \subseteq V_i$ for each $f \in C(R; V)$. Further, if f_i denotes the restriction of f to V_i then the map $\phi: C(R; V) \to C(S_1; V_1) \oplus \cdots \oplus C(S_t; V_t)$ given by $\phi(f) = \langle f_1, \ldots, f_t \rangle$ is a near-ring homomorphism. The map is onto, for if $\langle f_1, \ldots, f_t \rangle$ is in $C(S_1; V_1) \oplus \cdots \oplus C(S_t; V_t)$ extend each f_i to all of V by $\bar{f}_i(v_1 + \cdots + v_t) = f_i(v_i)$. Then $f = \sum \bar{f}_i$ is an element of $C(R; V)$ such that $\phi(f) = \langle f_1, \ldots, f_t \rangle$. To

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show that Φ is one-to-one we note that $e_i f(v_1 + \cdots + v_t) = f(e_i v_i) = f(v_i)$, $i = 1, \ldots, t$, implies $f(v_1 + \cdots + v_t) = f(v_1) + \cdots + f(v_t) = f_1(v_1) + \cdots + f_t(v_t)$. Hence if $\phi(f) = 0$ then $f = 0$. Therefore ϕ is an isomorphism and from Theorem 1 of [5] each $C(S_i; V_i)$ is a simple near-ring.

A type of converse to the proposition is also true. If $C(R; V)$ is a semisimple near-ring for every R-module V then in particular $C(R; R)$ is semisimple. But $C(R; R)$ is anti-isomorphic to R so R is a semisimple ring.

Again using Theorem 1 of [5] if $R = S_1 \oplus \cdots \oplus S_t$, S_i simple and not a field, or S_i is a field and $\dim_{S_i}(V_i) = 1$, we have $C(R; V)$ is a semisimple *ring*. Moreover, in this setting, $C(R; V) = \mathrm{End}_R(V)$. (See proof of Theorem 1 of [5].)

It is the goal of this paper to consider the following questions which arise naturally from the above remarks.

A. Which finite rings R have the property that $C(R; V)$ is a ring for every R-module V?

B. If $C(R; V)$ is a semisimple ring when is $C(R; V) = \mathrm{End}_R(V)$?

C. Which semisimple near-rings have the form $C(R; V)$ for some pair (R, V)?

In the next section we answer question A. In §3 we show that if $C(R; V)$ is a semisimple ring then one always has $C(R; V) = \mathrm{End}_R(V)$. Moreover if $C(R; V)$ is semisimple then information about the structure of the simple components is obtained, giving a partial answer to question C.

2. Strongly noncommutative rings. In this section we characterize those rings R such that $C(R; V)$ is a ring for every V. Recall that if R is a finite ring with identity then $R = T + M$ where $T \cap M = (0)$, M is a subgroup of rad R and $T = T_1 \oplus \cdots \oplus T_t$, T_i a complete $n_i \times n_i$ matrix ring over a local ring L_i with $T/\mathrm{rad}\, T \cong R/\mathrm{rad}\, R$ [7, p. 162]. Moreover there exist mutually orthogonal idempotents $e_1, \ldots, e_t$ in R such that $1 = e_1 + \cdots + e_t$ and $T_i = e_i R e_i$ for each i. Also $R/\mathrm{rad}\, R = S_1 \oplus \cdots \oplus S_t$ where each S_i is an $n_i \times n_i$ simple matrix ring and T_i is mapped onto S_i under the natural homomorphism $R \to R/\mathrm{rad}\, R$ (see [7, p. 162–163]). We say R is *strongly noncommutative* if $n_i > 1$ for $i = 1, 2, \ldots, t$.

THEOREM 2.1. *For a finite ring R with identity the following are equivalent*:

(i) *$C(R; V)$ is a ring for every faithful R-module V*;

(ii) *$C(R; V) = \mathrm{End}_R(V)$ for every faithful R-module V*;

(iii) *R is strongly noncommutative.*

PROOF. Since (ii) implies (i) is clear it remains to show (iii) implies (ii) and (i) implies (iii).

Suppose R is strongly noncommutative where, as above, $R = T + M$, $T = T_1 \oplus \cdots \oplus T_t$ with each T_i an $n_i \times n_i$ matrix ring over a local ring L_i and $n_i > 1$ for each i. If V is a faithful R-module then V is a faithful, unital T-module and $C(R; V) \subseteq C(T; V)$. Thus it suffices to show that for each $f \in C(T; V)$ and for each $v, w \in V$, $f(v + w) = f(v) + f(w)$. To this end let e_i be the identity for T_i; then $V = V_1 \oplus \cdots \oplus V_t$ where $V_i = e_i V$. We have $f(v_1 + \cdots + v_t) = f(v_1) + \cdots + f(v_t)$, $v_i \in V_i$, so it suffices to show $f(v_i^1 + v_i^2) = f(v_i^1) + f(v_i^2)$ for every $v_i^1, v_i^2 \in V_i$. Since $f(V_i) \subseteq V_i$, $f|V_i$ belongs to $C(T_i; V_i)$. Using an argument almost identical to the proof of Theorem 1 of [5], it is seen that $f|V_i$ is linear since $n_i > 1$.

Assume now that $C(R; V)$ is a ring for each R-module V but R is not strongly noncommutative. Then in the decomposition $R = T_1 \oplus \cdots \oplus T_t + M$ at least one T_i is a local ring, say T_1. We know $R/\text{rad } R \cong K_1 \oplus S_2 \oplus \cdots \oplus S_t$ where K_1 is a field and each S_i is a simple ring. Also under the homomorphism $R \to R/\text{rad } R$, $T_1 \to K_1$, $T_2 \to S_2, \ldots, T_t \to S_t$. Thus there exists a maximal ideal I containing $T_2, T_3, \ldots, T_t$ and rad R such that $R/I \cong K_1$. Under the action $r\bar{k} = \overline{rk}$, R/I is an irreducible R-module. Also $V = R \oplus R/I \oplus R/I$ is a faithful R-module under componentwise action. If we let $W = R/I \oplus R/I$ then $C(R; W)$ can be embedded in $C(R; V)$ as follows. For $\hat{g} \in C(R; W)$, define g: $V \to V$ by $g(r + \bar{k}_1 + \bar{k}_2) = \hat{g}(\bar{k}_1 + \bar{k}_2)$. We note further that since R/I is a field, $\text{Ann}_R(W) = I$ and so $C(R; W) \cong C(R/I; W) \cong C(K_1; W)$. Since $\dim_{K_1} W = 2$, it follows from Theorem 1 of [5] that $C(K_1; W)$ and hence $C(R; W)$ are not rings. Consequently $C(R; V)$ is not a ring, a contradiction. Thus it must be the case that R is strongly noncommutative.

For any finite ring R there exists an R-module V such that $C(R; V)$ is a ring; e.g., let $V = {}_RR$. Moreover it is always the case that $\text{End}_R(V) \subseteq C(R; V)$. We now give an example to show that it is possible for $C(R; V)$ to be a ring and yet $C(R; V) \neq \text{End}_R(V)$.

EXAMPLE 2.1. Let R be the ring consisting of the 3×3 matrices of the form

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \qquad a, b, c \in GF(2).$$

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Big| x, y, z \in GF(2) \right\}.$$

A calculation shows that $\text{End}_R(V) = R$. Another calculation gives $f(Rv) \subseteq Rv$ for each $f \in C(R; V)$ and for each $v \in V$. From this it follows that $C(R; V)$ is a ring since if $v \in V$ then

$$f(g + h)v = f(gv + hv) = f(r_1v + r_2v) = (r_1 + r_2)f(v) = (fg + fh)v.$$

Let $\{e_1, e_2, e_3\}$ be the standard basis for the vector space V over $GF(2)$. Then $V = R(e_1 + e_2 + e_3) \cup Re_2 \cup Re_3$ and the relation $f(e_1 + e_2 + e_3) = f(e_2) = f(e_3) = e_1$ determines a function in $C(R; V)$. But f is not in $\text{End}_R(V)$ since $f(e_2 + e_3) \neq f(e_2) + f(e_3)$. Hence $\text{End}_R(V) \neq C(R; V)$.

In the next section we show that if $C(R; V)$ is a *semisimple* ring then $C(R; V) = \text{End}_R(V)$.

3. Semisimple centralizer near-rings. Let $C(R; V)$ be semisimple. Then the center of $C(R; V)$ cannot contain nonzero nilpotent elements. Hence the center of R cannot contain nilpotent elements so the center of R is a direct sum of fields. Thus if n is the characteristic of R, we have $n = p_1p_2 \cdots p_s$ where the p_i's are distinct primes. But this implies that $R = R_1 \oplus \cdots \oplus R_s$ where R_i has characteristic p_i. Because it has characteristic p_i, R_i is an algebra over the field $GF(p_i)$ and so the

Wedderburn principal theorem [7, p. 164] holds for R_i. Consequently $R = \Sigma_{i,j} \oplus S_{ij} + N$ where each S_{ij} is a simple ring and N is a nilpotent ideal of R.

The following example shows that there exist semisimple centralizer near-rings that are not rings.

EXAMPLE 3.1. Let $R = \bar{R} \oplus F$ where $F = GF(2)$ and $\bar{R}$ is the simple ring of 2×2 matrices over $GF(2)$. Let $V_i = \{\binom{x}{y} | x, y \in F\}$, $i = 1, 2$, and let R act on $V = V_1 \oplus V_2$ componentwise. Then $C(R; V) \cong C(\bar{R}; V_1) \oplus C(F; V_2)$ where $C(\bar{R}; V_1)$ is a simple ring while $C(F; V_2)$ is a simple near-ring which is not a ring. Hence $C(R; V)$ is semisimple and not a ring.

For the remainder of this section we assume $C(R; V)$ is semisimple and investigate when $C(R; V)$ equals $\text{End}_R(V)$. As we have seen $R = S_1 \oplus \cdots \oplus S_t + N$ where each S_i is simple and N is a nilpotent ideal of R. We may assume $N \neq (0)$; otherwise R is semisimple and the results of §1 apply.

Assume $t = 1$, i.e. $R = S_1 + N$. From the proof of Lemma 1 of [5] it follows that $C(R; V)$ contains a function f such that $g_1 f g_2 f = 0$ for all $g_1, g_2 \in C(R; V)$. Hence $C(R; V)$ contains a nilpotent $C(R; V)$-subgroup and is not semisimple. So we may assume $t > 1$.

Let e_i denote the identity for S_i. Then $V = V_1 \oplus \cdots \oplus V_t$ where $V_i = \{v \in V | e_i v = v\}$. Also for $i, j = 1, 2, \ldots, t$ let $N_{ij} = e_i N e_j$. Then $N = \Sigma N_{ij}$. For $i = 1, \ldots, t$ let $B_i = \{w_i \in V_i | w_i = n_{ij} v_j$ for some $j \neq i$, $n_{ij} \in N_{ij}$, $v_j \in V_j\}$, and let W denote the subgroup of V generated by $B_1 \cup B_2 \cup \cdots \cup B_t$. Finally let $W_L = \{w \in V | f(w + v) = f(w) + f(v)$ for all $v \in V, f \in C(R; V)\}$.

LEMMA 3.1. *W is an R-submodule of V, W_L is a subgroup of V and $W \subseteq W_L$.*

PROOF. An element of W has the form $w = \Sigma n_{ij} v_j$ with $i \neq j$. For $n_{kl} \in N_{kl}$ and $n_{ij} v_j \in B_j$ we have $n_{kl} n_{ij} v_j \in B_k$ if $k \neq j$ and $n_{kl} n_{ij} v_j = n_{kl}(n_{ij} v_j) \in B_k$ if $k = j$. In this manner it is seen that $NW \subseteq W$. Also if $s \in S_1 \oplus \cdots \oplus S_t$ then $s n_{ij} v_j = (s n_{ij}) v_j \in B_i$ since $s n_{ij} \in N_{ij}$. Hence $SW \subseteq W$ and W is an R-submodule of V.

The second part of the lemma is straightforward and is omitted. To prove the last part if suffices to show that $B_i \subseteq W_L$ for each i. To this end let $v_i = n_{ij} v_j \in B_i$, $f \in C(R; V)$. For $k \neq i$ we have $f(v_i + v_k) = f(v_i) + f(v_k)$. For $v_i' \in V_i$,

$$f(v_i + v_i') = f(n_{ij} v_j + v_i') = f((n_{ij} + e_j)(v_j + v_i'))$$
$$= (n_{ij} + e_j) f(v_j + v_i') = (n_{ij} + e_j)[f(v_j) + f(v_i)] = f(v_i) + f(v_i').$$

With this it is easy to see that $f(v_i + v) = f(v_i) + f(v)$ for all $v \in V$, as desired.

From the lemma, every $f \in C(R; V)$ is linear on W and moreover $f(W) \subseteq W$. Suppose now that $C(R; V)$ is simple. Then the map $f \to f|W$ is an imbedding of $C(R; V)$ into $\text{End}_R(W)$. Also $W \neq (0)$, for otherwise $N_{ij} V_j = (0)$ for each $i \neq j$ and so each V_i is an R-module and $C(R; V)$-invariant. Hence $C(R; V)$ would not be simple. Thus $W \neq 0$ and $C(R; V)$ is a ring. This provides an alternate proof to Theorem 2 of [5].

LEMMA 3.2. *If the simple ring S_i is not a field then every $f \in C(R; V)$ is linear on V_i.*

PROOF. Again the restriction map $f \to f|V_i$ is a homomorphism of $C(R; V)$ into $C(S_i; V_i)$. Since $C(S_i; V_i) = \text{End}_{S_i}(V_i)$, every $f \in C(R; V)$ is linear on V_i.

Let v_i be a nonzero element in V_i. Then from the chain of S_i-submodules of V_i,

$$(0) \subseteq \ker N \cap V_i \subseteq \ker N^2 \cap V_i \subseteq \cdots \subseteq \ker N^{k-1} \cap V_i \subseteq V_i,$$

we see that there exists a unique integer $\rho(v_i)$ such that $v_i \in \ker N^{\rho(v_i)} \cap V_i$ but $v_i \notin \ker N^{\rho(v_i)-1} \cap V_i$. We call $\rho(v_i)$ the rank of v_i. For completeness let 0 have rank 0. We note that for v_i, v_i' in V_i we have $\rho(v_i + v_i') \leqslant \max\{\rho(v_i), \rho(v_i')\}$.

LEMMA 3.3. *If* $\ker N \cap V_i = \{0\}$ *then every* $f \in C(R; V)$ *is linear on* V_i.

PROOF. Assume $f \in C(R; V)$ such that f is not linear on V_i. Then there exist v_i, v_i' in V_i such that $f(v_i + v_i') - f(v_i) - f(v_i') \neq 0$. Among all such pairs $\{v_i, v_i'\}$ select one pair having an element of minimal rank, say $\{x_i, x_i'\}$, where x_i has minimal rank. For each $n_{ji} \in N_{ji}$ where $j \neq i$ we have $n_{ji}(f(x_i + x_i') - f(x_i) - f(x_i')) = 0$, since $n_{ji}x_i \in W$. Due to the minimality of x_i we also have

$$n_{ii}(f(x_i + x_i') - f(x_i) - f(x_i')) = 0$$

for each $n_{ii} \in N_{ii}$. Hence $f(x_i + x_i') - f(x_i) - f(x_i') \in \ker N \cap V_i$, a contradiction.

THEOREM 3.1. *Let* $C(R; V)$ *be a semisimple near-ring where* R *is not semisimple. Then* $R = S_1 \oplus \cdots \oplus S_t + N$ *where* $t > 1$, *each* S_i *is a simple ring and* N *is a nonzero nilpotent ideal of* R. *Moreover the following are equivalent.*

(i) $C(R; V)$ *is a ring.*

(ii) $C(R; V) = \text{End}_R(V)$.

(iii) *For each* i *at least one of the following is true*:

(a) S_i *is not a field*;

(b) S_i *is a field and* $\dim_{S_i}[\ker N \cap V_i] \leqslant 1$;

(c) $V_i \subseteq W$.

PROOF. The first part of the theorem has already been established. For the equivalences we start with (iii) $\to$ (ii). From Lemma 3.2 if S_i is not a field then every $f \in C(R; V)$ is linear on V_i. The same conclusion is true if $V_i \subseteq W$. So we may assume that at least one S_i is a field, say S_1, with $\dim_{S_1}[\ker N \cap V_1] \leqslant 1$ and $V_1 \nsubseteq W$. If $\ker N \cap V_1 = (0)$ then Lemma 3.3 applies. Therefore, we may also assume $\ker N \cap V_1$ is a 1-dimensional vector space over S_1.

Let $W_1 = W \cap V_1$ and $S = S_1 \oplus \cdots \oplus S_t$. V is a completely reducible S-module and we have, as S-modules, $V = \overline{V}_1 \oplus W_1 \oplus X$ where $X = V_2 \oplus \cdots \oplus V_t$, and $V_1 = \overline{V}_1 \oplus W_1$. Note that $W_1 \oplus X$ is an R-module and is $C(R; V)$-invariant. We select an S_1-basis $\{v_1, v_2, \ldots, v_l, w_1, \ldots, w_m\}$ for $\overline{V}_1 \oplus W_1$ as follows. Let $\{w_1, \ldots, w_m\}$ be a basis for W_1. Let $\{v_{t_1}, \ldots, v_l\}$ be a basis for $\overline{V}_1^1 = \{v \in \overline{V}_1 | N_{11}v \subseteq W_1\}$. Let $\{v_{t_2}, \ldots, v_{t_1}, \ldots, v_l\}$ be a basis for $\overline{V}_1^2 = \{v \in \overline{V}_1 | N_{11}v \subseteq \overline{V}_1^1\}$, etc. Using the fact that N_{11} is nilpotent, this process terminates to give the desired basis $\{v_1, \ldots, v_l, w_1, \ldots, w_m\}$ for $\overline{V}_1 \oplus W_1$. Thus every $v \in V$ can be uniquely represented in the form $v = s_{11}v_1 + \cdots + s_{1l}v_l + w + x$ where $s_{1i} \in S_1$, $w \in W_1$, $x \in X$.

Let k be a nonzero element in $\ker N \cap V_1$. The function f: $V \to V$ defined by

$f(s_{11}v_1 + \cdots + s_{1l}v_l + w + x) = s_{11}k$ belongs to $C(R; V)$. Let $L = C(R; V)f$, the $C(R; V)$-subgroup generated by f. If $k \in W_1$ then $g(k) \in \ker N \cap W_1$ for each $g \in C(R; V)$, and thus $g_1 f g_2 f = 0$. Thus $L^2 = (0)$, a contradiction to $C(R; V)$ being semisimple. Hence $\ker N \cap W_1 = (0)$ and, since $\bar{V}_1$ was an arbitrary complement of W_1 in V_1, we may reselect $\bar{V}_1$ if necessary such that $\ker N \cap V_1 \subseteq \bar{V}_1$; i.e. $V_1 = \tilde{V}_1 \oplus (\ker N \cap V_1) \oplus W_1$ where $\bar{V}_1 = \tilde{V}_1 \oplus (\ker N \cap V_1)$. If $\tilde{V}_1 \neq (0)$ then we may assume our first basis element v_1 belongs to $\tilde{V}_1$. But once again, if f is defined as above, we get $L^2 = (0)$. Hence $\tilde{V}_1 = (0)$ and $\bar{V}_1 = \ker N \cap V_1$. We now have $V = (\ker N \cap V_1) \oplus W_1 \oplus X$. Since

$$\dim_{S_1} (\ker N \cap V_1) = 1,$$

each $f \in C(R; V)$ is trivially linear on $\ker N \cap V_1$ and hence on all of V_1. This shows that (iii) $\rightarrow$ (ii).

Suppose (i) is true. Then we may assume by way of contradiction that some S_i is a field, say S_1, that $\dim_{S_1}[\ker N \cap V_1] > 1$ and that $V_1 \not\subseteq W$. Because $C(R; V)$ is semisimple the arguments above imply $V = (\ker N \cap V_1) \oplus W_1 \oplus X$ where W_1 and X are defined as before. But $\ker N \cap V_1$ and $W_1 \oplus X$ are both R-modules and both $C(R; V)$-invariant. Hence

$$C(R; V) \cong C(S_1; \ker N \cap V_1) \oplus C(R; W_1 \oplus X).$$

Since $\dim_{S_1}(\ker N \cap V_1) > 1$, the first summand is not a ring. Hence (i) $\rightarrow$ (iii). Since (ii) $\rightarrow$ (i) is obvious the proof is complete.

As a consequence of this theorem we note that if $C(R; V)$ is a simple ring where R is not a field then $C(R; V) = \mathrm{End}_R(V)$. This was stated as Theorem 3 in [5] but the proof given there is incorrect.

We also note that as a consequence of the proof of Theorem 3.1 and the preliminaries to it we have the following structural result for semisimple near-rings of the form $C(R; V)$.

Corollary. *If $C(R; V)$ is semisimple then $C(R; V) = A_1 \oplus \cdots \oplus A_t$ where each A_i is either a simple ring or a simple near-ring of the form $C(F_i; V_i)$ where V_i is a vector space over a field F_i. Moreover if R is not semisimple then at least one A_i must be a ring.*

Proof. It remains to prove the last part of the corollary. Since $C(R; V)$ is semisimple then $R = S_1 \oplus \cdots \oplus S_k + N$ where $N = \mathrm{rad}\, R$ and each S_i is simple with identity e_i. As before let $N_{ij} = e_i N e_j$ and let W be the R-submodule of V as in Lemma 3.1. If $W = (0)$ then $N_{ij} V_j = (0)$ for each $i \neq j$ where V_j is the 1-space for e_j. This means each V_i is an R-module as well as $C(R; V)$-invariant. Hence

$$C(R; V) \cong C(R_1; V_1) \oplus \cdots \oplus C(R_k; V_k)$$

where $R_i = S_i + N_{ii}$. Since $C(R; V)$ is semisimple each $C(R_i; V_i)$ is semisimple [8, p. 146]. We show now that if $N_{ii} \neq (0)$ then $C(R_i; V_i)$ cannot be semisimple. Suppose $N_{ii}^l = (0)$ but $N_{ii}^{l-1} \neq (0)$. Let $W_1 = \ker N_{ii}^{l-1} = \{v \in V_i | nv = 0$ for all $n \in N_{ii}^{l-1}\}$, a proper subgroup of V_i, an S_i-submodule, and $C(R_i; V_i)$-invariant. As an S_i-module V_i is completely reducible so $V_i = W_1 \oplus W_2$, an S_i-module direct sum. As constructed in the proof of Lemma 1 of [5] there exists a nonzero $f \in C(R_i; V_i)$ such that $f(V_i) \subseteq W_1$ and $f(W_1) = \{0\}$. Let $I = \{f \in C(R_i; V_i) | f(V_i) \subseteq W_1$ and $f(W_1) = \{0\}\}$. Then I is a nilpotent $C(R_i; V_i)$-subgroup ($I^2 = (0)$) and hence $C(R_i; V_i)$ is not semisimple. So each $N_{ii} = (0)$ and since $N_{ij}V = (0)$, $N_{ij} = (0)$ if $i \neq j$. Thus R is semisimple.

So we may assume $W \neq (0)$. Since W is $C(R; V)$-invariant the map $f \to f|W$ is a homomorphism of $C(R; V)$ into the ring $\mathrm{End}_R(W)$. Hence a nontrivial homomorphic image of $C(R; V)$ is a ring and this implies at least one simple component of $C(R; V)$ is a ring [8, p. 55].

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Department of Mathematics, Texas A&M University, College Station, Texas 77843