PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, October 1980

ON OPENNESS OF H_n-LOCUS AND SEMICONTINUITY OF nTH DEVIATION

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ABSTRACT. M. André has used the property H_n, namely the vanishing of certain homology groups, and the deviation δ_n to characterize some classes of rings. In the present paper the author establishes an inequality on the deviations and obtains a Nagata criterion for H_n-locus and its openness for quotients of complete intersection rings and excellent rings. The upper-semicontinuity for δ_n is also proved for the same classes of rings.

Introduction. We study a property of local rings $(A, \mathfrak{m}, K)$ introduced by M. André, namely the vanishing of $H_n(A, K, K)$ which gives regularity (resp. complete intersection) for $n = 2$ (resp. $n = 3$).

At first we prove (Theorem 1.7) a property for the deviations δ_n introduced by M. André which resembles a result of L. L. Avramov (cf. [**Av**]) on deviations ε_n which appear in [**G-L**]. By an inequality on these deviations δ_n (Theorem 2.3) we can prove a Nagata criterion for H_n-locus and its openness for some class of rings (Corollaries 3.5 and 3.7). Then we obtain upper-semicontinuity for δ_n on excellent rings, for $n \neq 1$, and on quotients of complete intersection rings, for $n > 3$. The previous inequality becomes equality in some particular cases (Propositions 4.1 and 4.2), so we can prove that δ_n is constant on locally closed sets on every quotient of complete intersection rings.

We conclude this paper showing the openness of $\delta_3(1)$-locus on a class of rings, where $\delta_3(1)$ is a property "near" to complete intersection, and a sort of Nagata criterion for δ_n upper-semicontinuous.

We wish to thank Silvio Greco for his useful suggestions on this paper.

1. All rings considered in this paper are unitary, commutative and noetherian. A local ring A with maximal ideal $\mathfrak{m}$ and residue field K will be often denoted by $(A, \mathfrak{m}, K)$.

Let $\mathbf{P}$ be a property of local rings, the $\mathbf{P}$-locus of a local ring A is the set $U_{\mathbf{P}}(A)$ consisting of those $\mathfrak{p} \in \operatorname{spec}(A)$ such that $A_{\mathfrak{p}}$ has property $\mathbf{P}$. We want to prove the openness of $U_{\mathbf{P}}(A)$ for some homological property $\mathbf{P}$. First we recall a few definitions that will be used in the following.

Received by the editors March 12, 1979 and, in revised form, July 31, 1979.
1980 *Mathematics Subject Classification*. Primary 13D99.
[1]The present paper was written while the author was supported by a C.N.R.-N.A.T.O. fellowship at Brandeis University.

DEFINITION 1.1. *A local ring A is called a complete intersection* (*C.I.*) *iff its* $\mathfrak{m}$-*adic completion* $\hat{A}$ ($\mathfrak{m}$ *denotes the maximal ideal of* A) *is a quotient of a regular local ring by a regular sequence or equivalently* $\hat{A} = R/\mathfrak{a}$ *with* R *a regular local ring and* $\mathrm{ht}(\mathfrak{a}) = \mu(\mathfrak{a})$, *where* ht *means "height" and* μ *means "minimal # of generators".*

DEFINITION 1.2. *A ring A is called a* (*locally*) *complete intersection iff* $A_{\mathfrak{p}}$ *is C.I. for every* $\mathfrak{p} \in \mathrm{spec}(A)$.

DEFINITION 1.3. *A local ring* $(A, \mathfrak{m}, K)$ *such that* $H_n(A, K, K) = 0$ *will be called an* H_n *ring.*

DEFINITION 1.4. *A ring A will be called locally or strongly* H_n *iff for every* $\mathfrak{p} \in \mathrm{spec}(A)$ $A_{\mathfrak{p}}$ *is* H_n.

DEFINITION 1.5. *For any local ring* $(A, \mathfrak{m}, K)$ *we call the nth deviation the integer*

$$\delta_n(A) = \dim_K H_n(A, K, K).$$

For the definition of the nth homology module $H_n(A, B, W)$ and for the principal properties of H_n ring we refer to [$\mathbf{A}_1$] and [$\mathbf{A}_2$].

Now we prove a theorem that leads us to certain classes of rings in which H_n is equivalent to strongly H_n. First we need a lemma.

LEMMA 1.6. *Let* $(A, \mathfrak{m}, K)$ *be a local ring, quotient of a C.I. ring, and* $\hat{A}$ *its* $\mathfrak{m}$-*adic completion. Then* $H_n(A, \hat{A}, W) = 0$ *for all* $n \geqslant 3$ *and every A-module W.*

PROOF. Suppose B is the given C.I. ring, since $\hat{B}$ is B-flat, by [$\mathbf{A}_1$, Proposition IV.54], we have

$$H_n(B, \hat{B}, W) = H_n(A, \hat{B} \otimes_B A, W) = H_n(A, \hat{A}, W)$$

for any n and any $\hat{A}$-module W. Take $\mathfrak{q} \in \mathrm{spec}(\hat{B})$ and $\mathfrak{p} = B \cap \mathfrak{q}$, $\hat{B}_{\mathfrak{q}}$ and $B_{\mathfrak{p}}$ are still a C.I. (cf. [Av, Proposition 3.8]), so

$$H_n(B_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{p})) = H_n(\hat{B}_{\mathfrak{q}}, k(\mathfrak{q}), k(\mathfrak{q})) = 0$$

for $n \geqslant 3$ (see [$\mathbf{A}_1$, Propositions VI.27 and X.20]). Now from the ring homomorphisms $B_{\mathfrak{p}} \to k(\mathfrak{p}) \to k(\mathfrak{q})$ we have the exact sequence

$$H_n(B_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{q})) \to H_n(B_{\mathfrak{p}}, k(\mathfrak{q}), k(\mathfrak{q})) \to H_n(k(\mathfrak{p}), k(\mathfrak{q}), k(\mathfrak{q}))$$

but

$$H_n(B_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{q})) = H_n(B_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} k(\mathfrak{q}) = 0$$

and $H_n(k(\mathfrak{p}), k(\mathfrak{q}), k(\mathfrak{q})) = 0$ (cf. [$\mathbf{A}_1$, Propositions III.20 and VII.4]) so

$$H_n(B_{\mathfrak{p}}, k(\mathfrak{q}), k(\mathfrak{q})) = 0.$$

Now from $B_{\mathfrak{p}} \to \hat{B}_{\mathfrak{q}} \to k(\mathfrak{q})$ we get

$$H_{n+1}(\hat{B}_{\mathfrak{q}}, k(\mathfrak{q}), k(\mathfrak{q})) \to H_n(B_{\mathfrak{p}}, \hat{B}_{\mathfrak{q}}, k(\mathfrak{q})) \to H_n(B_{\mathfrak{p}}, k(\mathfrak{q}), k(\mathfrak{q}))$$

so

$$H_n(B_{\mathfrak{p}}, \hat{B}_{\mathfrak{q}}, k(\mathfrak{q})) = H_n(B, \hat{B}, k(\mathfrak{q})) = 0.$$

The proof follows from [$\mathbf{A}_1$, Supplement Proposition 29].

THEOREM 1.7. *Let $(A, \mathfrak{m}, K)$ be a local ring, $\mathfrak{p}$ any prime ideal and n an integer. If one of the following conditions is satisfied*:

(a) *A is pseudo-geometric and $n \geqslant 2$*,

(b) *A is a quotient of a C.I. and $n \geqslant 3$*,

then

$$\delta_n(A_\mathfrak{p}) \leqslant \delta_n(A).$$

PROOF. (a) We can suppose A to be henselian, in fact the hypotheses are preserved by henselization, on the other hand from

$$A \to {}^hA \to K$$

we obtain

$$H_n(A, {}^hA, K) \to H_n(A, K, K) \to H_n({}^hA, K, K) \to H_{n-1}(A, {}^hA, K)$$

and because of absolute flatness of the A-module ${}^hA$, by [$\mathbf{A_1}$, Proposition V.25], we will have $\delta_n(A) = \delta_n({}^hA)$; similarly for every $\mathfrak{p} \in \operatorname{spec}(A)$ and $\mathfrak{p}' \in \operatorname{spec}({}^hA)$, such that $\mathfrak{p}' \cap A = \mathfrak{p}$, we have $\delta_n(A_\mathfrak{p}) = \delta_n[({}^hA)_{\mathfrak{p}'}]$.

Now take $\mathfrak{p} \in \operatorname{spec}(A)$ such that $\dim A/\mathfrak{p} = 1$ and let B be the integral closure of $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is a henselian ring, by hypothesis we find out that B is a DVR. Let $\mathfrak{n} = (t)$ be the maximal ideal of B and $L = B/\mathfrak{n}$; we have

$$\begin{aligned}\delta_n(A_\mathfrak{p}) &= \dim_{k(\mathfrak{p})} H_n(A_\mathfrak{p}, k(\mathfrak{p}), k(\mathfrak{p})) = \dim_{k(\mathfrak{p})} H_n(A, B, B) \otimes_B k(\mathfrak{p})\\ &\leqslant \text{minimal \# of generators of the } B\text{-module } H_n(A, B, B)\\ &= \dim_L H_n(A, B, B)/tH_n(A, B, B).\end{aligned}$$

From the exact sequence $0 \to B \xrightarrow{t} B \to L \to 0$ we have

$$H_n(A, B, B) \xrightarrow{t} H_n(A, B, B) \to H_n(A, B, L)$$

exact, by [$\mathbf{A_1}$, III.22], so

$$\delta_n(A_\mathfrak{p}) \leqslant \dim_L H_n(A, B, L).$$

Now from $A \to B \to L$, by using the homology sequence

$$H_{n+1}(B, L, L) \to H_n(A, B, L) \to H_n(A, L, L) \to H_n(B, L, L)$$

and regularity of B, i.e. $H_n(B, L, L) = 0$ for $n \geqslant 2$, we get $H_n(A, B, L) = H_n(A, L, L)$.

On the other hand, from $A \to K \to L$ and its homology sequence, we can obtain

$$H_n(A, L, L) = H_n(A, K, L) = H_n(A, K, K) \otimes_K L.$$

Finally

$$\delta_n(A_\mathfrak{p}) \leqslant \dim_L H_n(A, B, L) = \dim_K H_n(A, K, K) = \delta_n(A).$$

Now, for every $\mathfrak{p} \in \operatorname{spec}(A)$, the proof follows by induction on $\dim A/\mathfrak{p}$.

(b) We can suppose A to be a complete local ring, in fact the hypotheses are trivially preserved by completion, furthermore $\delta_n(A) = \delta_n(\hat{A})$ (cf. [$\mathbf{A_1}$, Proposition X.18]) and, if $\mathfrak{p} \in \operatorname{spec}(A)$ and $\mathfrak{p}' \in \operatorname{spec}(\hat{A})$ are such that $\mathfrak{p}' \cap A = \mathfrak{p}$, from $A \to \hat{A} \to k(\mathfrak{p}')$ we have

$$H_n(A, \hat{A}, k(\mathfrak{p}')) \to H_n(A, k(\mathfrak{p}'), k(\mathfrak{p}')) \to H_n(\hat{A}, k(\mathfrak{p}'), k(\mathfrak{p}'))$$

and by Lemma 1.6

$$\dim_{k(\mathfrak{p}')} H_n(A, k(\mathfrak{p}'), k(\mathfrak{p}')) \leqslant \dim_{k(\mathfrak{p}')} H_n(\hat{A}, k(\mathfrak{p}'), k(\mathfrak{p}'))$$

so

$$\begin{aligned}\delta_n(A_{\mathfrak{p}}) &= \dim_{k(\mathfrak{p})} H_n(A_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{p})) = \dim_{k(\mathfrak{p}')} H_n(A, k(\mathfrak{p}), k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} k(\mathfrak{p}') \\ &= \dim_{k(\mathfrak{p}')} H_n(A, k(\mathfrak{p}'), k(\mathfrak{p}')) \leqslant \dim_{k(\mathfrak{p}')} H_n(\hat{A}, k(\mathfrak{p}'), k(\mathfrak{p}')) = \delta_n(\hat{A}_{\mathfrak{p}'}).\end{aligned}$$

Now we can use the same reasoning as in (a) because A will be henselian and pseudo-geometric (indeed excellent).

REMARK 1.8. By some result of André (cf. [$\mathbf{A_1}$, XIX.21, XX.26, XX.27]) the nth deviation $\delta_n(A)$ coincides with the $n-1$th "deviation" $\varepsilon_{n-1}(A)$ defined by Gulliksen and Levin in [G-L] for $n \leqslant \pi(A)$, where

$$\pi(A) = \begin{cases} \infty & \text{if } \operatorname{char}(K) = 0, \\ 2p & \text{if } \operatorname{char}(K) = p. \end{cases}$$

Then Theorem 1.7 is really different from [Av, Theorem 2.6], for $\operatorname{char}(K) \neq 0$ and suitable large n. So we can eliminate the hypotheses on n from the previous theorem: we need only $n > 0$.

COROLLARY 1.9. *Let $(A, \mathfrak{m}, K)$ be a local ring satisfying one of the following properties*:

(a) *A is pseudo-geometric (in particular excellent)*,

(b) *A is a quotient of a C.I.*,

then A is H_n iff it is strongly H_n, for any n.

PROOF. We must only say that for $n = 0$ the statement is vacuous (see [$\mathbf{A_1}$, Lemma 60]).

The proposition below is a generalization of Corollary 1.9(b).

PROPOSITION 1.10. *Let B be a local H_{n-1} and strongly H_n ring, $\mathfrak{b} \subseteq B$ any ideal. Then $A = B/\mathfrak{b}$ is H_n iff it is strongly H_n, for any integer n.*

PROOF. Put $K = A/\mathfrak{m}$, the residue field of A (hence of B), take the ring homomorphisms $B \to A \to K$; we have the exact sequence

$$H_n(A, K, K) \to H_{n-1}(B, A, K) \to H_{n-1}(B, K, K).$$

Now suppose A is an H_n ring, we find $H_{n-1}(B, A, K) = 0$ and for any $\mathfrak{p} \in \operatorname{spec}(A)$, by [$\mathbf{A_2}$, Proposition 27.7], $H_{n-1}(B, A, k(\mathfrak{p})) = 0$.

Now take $B \to A \to k(\mathfrak{p})$; we obtain the exact sequence

$$H_n(B, k(\mathfrak{p}), k(\mathfrak{p})) \to H_n(A, k(\mathfrak{p}), k(\mathfrak{p})) \to H_{n-1}(B, A, k(\mathfrak{p}))$$

but

$$H_n(B, k(\mathfrak{p}), k(\mathfrak{p})) = H_n(B_{\mathfrak{p}'}, k(\mathfrak{p}'), k(\mathfrak{p}')) = 0,$$

where $\mathfrak{p}' \in \operatorname{spec}(B)$ and $\mathfrak{p}'/\mathfrak{b} = \mathfrak{p}$, and the last equality holds since B is strongly H_n. So we can conclude

$$H_n(A_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{p})) = H_n(A, k(\mathfrak{p}), k(\mathfrak{p})) = 0.$$

2. In the proof of the main theorem on deviations we will use the following two lemmas.

LEMMA 2.1. *Let* $(A, \mathfrak{m}, K)$ *be a local domain, L its quotient field and M an A-free module, then* $\dim_K M \otimes_A K = \dim_L M \otimes_A L$.

PROOF. The proof is a straightforward computation.

LEMMA 2.2. *Let* $(A, \mathfrak{m}, K)$ *be a local ring, n any integer and* $\mathfrak{p} \in \operatorname{spec}(A)$ *such that* $H_i(A, A/\mathfrak{p}, A/\mathfrak{p})$ *is* $A/\mathfrak{p}$*-free for* $0 \leq i < n$. *Then*

$$H_n(A, A/\mathfrak{p}, K) = H_n(A, A/\mathfrak{p}, A/\mathfrak{p}) \otimes_{A/\mathfrak{p}} K.$$

PROOF. From the spectral sequence

$$\operatorname{Tor}_i^{A/\mathfrak{p}}(H_j(A, A/\mathfrak{p}, A/\mathfrak{p}), K) \Rightarrow H_{i+j}(A, A/\mathfrak{p}, K)$$

for $i + j = n$ we have

$$i > 0, j \leq n-1, \qquad \operatorname{Tor}_i^{A/\mathfrak{p}}(H_j(A, A/\mathfrak{p}, A/\mathfrak{p}), K) = 0$$

$$\begin{aligned} i = 0, j = n, \qquad \operatorname{Tor}_0^{A/\mathfrak{p}}(H_n(A, A/\mathfrak{p}, A/\mathfrak{p}), K) &= H_n(A, A/\mathfrak{p}, A/\mathfrak{p}) \otimes_{A/\mathfrak{p}} K \\ &= H_n(A, A/\mathfrak{p}, K). \end{aligned}$$

THEOREM 2.3. *Let A be a ring and* $\mathfrak{p} \in \operatorname{spec}(A)$. *Then for any integer n there exists an open subset* U_n, $\mathfrak{p} \in U_n$, *such that, for every* $\mathfrak{q} \in U_n \cap V(\mathfrak{p})$, *we have*

$$\delta_n(A_\mathfrak{q}) \leq \delta_n(A_\mathfrak{p}) + \delta_n(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}). \tag{1}$$

PROOF. Since $H_i(A, A/\mathfrak{p}, A/\mathfrak{p})$ is $A/\mathfrak{p}$-finite for each i (cf. [$\mathbf{A_1}$, Proposition IV.55]), because of the theorem of generic flatness (cf. [**M**, Theorem 53]) there exists $f_i \notin \mathfrak{p}$ such that $H_i(A_{f_i}, A_{f_i}/\mathfrak{p}_{f_i}, A_{f_i}/\mathfrak{p}_{f_i})$ is free (cf. [**B**, Chapter II.5 # 1]).

Now we can shrink to the open neighbourhood of $\mathfrak{p}$, $A_{f_0 \cdot \ldots \cdot f_n}$, which from now on will be denoted by A, and then we have $H_i(A, A/\mathfrak{p}, A/\mathfrak{p})$ free for any i, $0 \leq i \leq n$. By localization in every $\mathfrak{q} \supseteq \mathfrak{p}$ of such an open set, we can assume $(A, \mathfrak{m}, K)$ to be a local ring and we have to prove

$$\delta_n(A) \leq \delta_n(A_\mathfrak{p}) + \delta_n(A/\mathfrak{p}).$$

Take the ring homomorphisms $A \to A/\mathfrak{p} \to K$; from the homology sequence

$$H_n(A, A/\mathfrak{p}, K) \to H_n(A, K, K) \to H_n(A/\mathfrak{p}, K, K)$$

we obtain

$$\dim_K H_n(A, K, K) \leq \dim_K H_n(A, A/\mathfrak{p}, K) + \dim_K H_n(A/\mathfrak{p}, K, K)$$

and by Lemma 2.2

$$\dim_K H_n(A, K, K) \leq \dim_K H_n(A, A/\mathfrak{p}, A/\mathfrak{p}) \otimes_{A/\mathfrak{p}} K + \dim_K H_n(A/\mathfrak{p}, K, K)$$

since $H_n(A, A/\mathfrak{p}, A/\mathfrak{p})$ is $A/\mathfrak{p}$-free. By Lemma 2.1 we get

$$\begin{aligned} \dim_K H_n(A, A/\mathfrak{p}, A/\mathfrak{p}) \otimes_{A/\mathfrak{p}} K &= \dim_{K(\mathfrak{p})} H_n(A, A/\mathfrak{p}, A/\mathfrak{p}) \otimes_{A/\mathfrak{p}} k(\mathfrak{p}) \\ &= \dim_{k(\mathfrak{p})} H_n(A, A/\mathfrak{p}, k(\mathfrak{p})) = \dim_{k(\mathfrak{p})} H_n(A_\mathfrak{p}, k(\mathfrak{p}), k(\mathfrak{p})) \end{aligned}$$

so we can conclude the proof.

3. Now we use the previous result to prove the openness of the H_n-locus and on the other hand the upper-semicontinuity of δ_n.

PROPOSITION 3.1 (NAGATA CRITERION FOR H_n-LOCUS). *Let A be a ring and n an integer such that $H_n \Leftrightarrow$ strongly H_n for any localization of A. If for every $\mathfrak{p} \in U_{H_n}(A)$, $U_{H_n}(A/\mathfrak{p})$ contains a nonempty open subset, then $U_{H_n}(A)$ is open.*

PROOF. By hypothesis on A and n, the H_n property is stable under generalizations so, by [**M**, 22 B, Lemma 2], we must show that $U_{H_n}(A) \cap V(\mathfrak{p})$ contains a nonempty open subset for any $\mathfrak{p} \in U_{H_n}(A)$.

Now since $U_{H_n}(A/\mathfrak{p})$ contains a nonempty open subset we may shrink to an open neighbourhood of $\mathfrak{p}$ such that $U_{H_n}(A/\mathfrak{p}) = \operatorname{spec}(A/\mathfrak{p})$ and $(A, \mathfrak{m}, K)$ is local. Because of Theorem 2.3 we can find another neighbourhood of $\mathfrak{p}$ in which

$$\delta_n(A) \leqslant \delta_n(A_\mathfrak{p}) + \delta_n(A/\mathfrak{p})$$

from which our contention follows.

REMARK 3.2. The hypotheses on A and n of the previous proposition hold for n an even integer such that $n \leqslant \inf\{\pi(A_\mathfrak{p}) | \mathfrak{p} \in \operatorname{spec}(A)\}$ (see again [**Av**, Theorem 2.6]). They hold also for A a quotient of a (locally) C.I. ring but in this case the openness of $U_{H_n}(A)$ is automatic, that is without further assumption on $U_{H_n}(A/\mathfrak{p})$, as we prove below.

LEMMA 3.3. *Let $(A, \mathfrak{m}, K)$ be a local ring, $\mathfrak{p} \in \operatorname{spec}(A)$. If*

$$H_i(A, A/\mathfrak{p}, A/\mathfrak{p}) = \begin{cases} 0 & \text{for } i = n, n-1, \\ \text{free} & \text{for } 0 \leqslant i < n-1, \end{cases}$$

then A is H_n iff $A/\mathfrak{p}$ is H_n.

PROOF. By Lemma 2.2 we have $H_i(A, A/\mathfrak{p}, K) = 0$ for $i = n$, $n-1$, so we conclude from the homology sequence

$$H_n(A, A/\mathfrak{p}, K) \to H_n(A, K, K) \to H_n(A/\mathfrak{p}, K, K) \to H_{n-1}(A, A/\mathfrak{p}, K).$$

PROPOSITION 3.4. *Let B be a locally H_n, H_{n-1} ring and $A = B/\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq B$, then $U_{H_n}(A)$ is open.*

PROOF. By hypothesis on B and by Proposition 1.6, A and n satisfy the conditions of Proposition 3.1, so it is enough to prove that for any $\mathfrak{p} \in U_{H_n}(A)$, $U_{H_n}(A/\mathfrak{p})$ contains a nonempty open subset to conclude our contention.

Put $\mathfrak{p}' \in \operatorname{spec}(B)$ such that $\mathfrak{p}'/\mathfrak{b} = \mathfrak{p}$, so that $A/\mathfrak{p} = B/\mathfrak{p}'$. We have

$$0 = H_i(B_{\mathfrak{p}'}, k(\mathfrak{p}'), k(\mathfrak{p}')) = H_i(B, B/\mathfrak{p}', B/\mathfrak{p}') \otimes_{B/\mathfrak{p}'} B_{\mathfrak{p}'}$$

for $i = n$, $n-1$; so we may shrink to an open neighbourhood of $\mathfrak{p}'$ in which we can suppose $H_i(B, B/\mathfrak{p}', B/\mathfrak{p}') = 0$ for $i = n$, $n-1$. Now by the same reasoning as in Theorem 2.3 we may shrink to an open neighbourhood of $\mathfrak{p}'$ in which $H_i(B, B/\mathfrak{p}', B/\mathfrak{p}')$ is free, for each $0 \leqslant i < n-1$, and we can suppose B local. So, since B is H_n, by Lemma 3.3, we find $B/\mathfrak{p}' = A/\mathfrak{p}$ is H_n.

COROLLARY 3.5. *Let A be a quotient of a (locally) C. I. ring (in particular regular), then $U_{H_n}(A)$ is open for any $n \geqslant 3$.*

PROOF. For $n = 3$ see [G-M, Corollary 3.4]; for $n > 3$ it is enough to remember that for any local ring strongly C.I.$\Leftrightarrow$C.I.$\Leftrightarrow H_i$ for all $i \geqslant 3$, and then to use Proposition 3.4.

From now on we will say that a local ring A has property $\Delta_n(s)$ if $\delta_n(A) \leqslant s$. So for any local ring A we put

$$U_{\Delta_n(s)}(A) = \{\mathfrak{p} \in \operatorname{spec}(A) | A_\mathfrak{p} \text{ has } \Delta_n(s), \text{ i.e. } \delta_n(A_\mathfrak{p}) \leqslant s\}.$$

PROPOSITION 3.6. *Let A be an excellent ring, then for every $n \neq 1$ δ_n is upper-semi-continuous on A, i.e. $U_{\Delta_n(s)}(A)$ is open for any integer s.*

PROOF. By Theorem 1.7 the property $\Delta_n(s)$ is stable under generalizations; we only have to show that for any $\mathfrak{p} \in U_{\Delta_n(s)}(A)$, $U_{\Delta_n(s)}(A) \cap V(\mathfrak{p})$ contains a non-empty open subset.

Take $\mathfrak{p} \in U_{\Delta_n(s)}(A)$; by Theorem 2.3 we can find a neighbourhood of $\mathfrak{p}$, U', such that for $\mathfrak{q}' \in U' \cap V(\mathfrak{p})$, (1) holds. Since A is excellent $U_{H_2}(A/\mathfrak{p})$ is open, so $U_{H_n}(A/\mathfrak{p})$ contains a nonempty open subset for each $n \neq 1$. Then there exists another neighbourhood of $\mathfrak{p}$, U'', in which for $\mathfrak{q}'' \in U'' \cap V(\mathfrak{p})$, $\delta_n(A_{\mathfrak{q}''}/\mathfrak{p}A_{\mathfrak{q}''}) = 0$, with $n \neq 1$.

Now for any $\mathfrak{q} \in U' \cap U'' \cap V(\mathfrak{p})$ we have

$$\delta_n(A_\mathfrak{q}) \leqslant \delta_n(A_\mathfrak{p}) \leqslant s.$$

COROLLARY 3.7. *Let A be an excellent ring, then $U_{H_n}(A)$ is open for $n \neq 1$.*

PROOF. Use Proposition 3.6 for $s = 0$.

PROPOSITION 3.8. *Let B be a (locally) C.I. ring and $\mathfrak{b} \subseteq B$ any ideal, then δ_n is upper-semicontinuous on $A = B/\mathfrak{b}$ for every $n \geqslant 3$.*

PROOF. As in Proposition 3.6, by Corollary 3.5.

REMARK 3.9. For A an excellent ring (or quotient of a locally C.I. ring) the function δ_n is (upper) limited for $n \neq 1$ (resp. $n \geqslant 3$). In fact

$$U_{\Delta_n(0)} \subseteq U_{\Delta_n(1)} \subseteq \cdots \subseteq U_{\Delta_n(s)} \subseteq \cdots$$

produces a nondecreasing chain of open sets, by Propositions 3.6 and 3.8, and by noetheriantity it is stable, so for a sufficiently large s, $U_{\Delta_n(s)} = U_{\Delta_n(s+1)} = \cdots$, i.e. $\delta_n(A_\mathfrak{p}) \leqslant s$ for every $\mathfrak{p} \in \operatorname{spec}(A)$.

4. At last we devote our attention to some particular cases in which (1) becomes an equality.

PROPOSITION 4.1. *Let A be a quotient of a (locally) C. I. ring B, then for any $\mathfrak{p} \in \operatorname{spec}(A)$ and $n > 3$ there is an open set U, $\mathfrak{p} \in U$, such that for every $\mathfrak{q} \in U \cap V(\mathfrak{p})$*

$$\delta_n(A_\mathfrak{q}) = \delta_n(A_\mathfrak{p}) + \delta_n(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}) \tag{2}$$

holds.

PROOF. Take $\mathfrak{p} \in \operatorname{spec}(A)$ and $n > 3$; by Corollary 3.5 $U_{H_{n-1}}(A)$ and $U_{H_n}(A/\mathfrak{p})$ are open, so we may shrink to an open neighbourhood of $\mathfrak{p}$ in which we can suppose $(A, \mathfrak{m}, K)$ local and $H_n(A/\mathfrak{p}, K, K) = H_{n-1}(A, K, K) = 0$; then from the homology sequence

$$H_n(A/\mathfrak{p}, K, K) \to H_{n-1}(A, A/\mathfrak{p}, K) \to H_{n-1}(A, K, K)$$

we have $H_{n-1}(A, A/\mathfrak{p}, K) = 0$.

Now let $\mathfrak{p}' \in \operatorname{spec}(B)$ be the prime ideal whose image in A is $\mathfrak{p}$; it will be

$$H_n(B, A/\mathfrak{p}, K) = H_n(B, A/\mathfrak{p}, k(\mathfrak{p})) = H_n(B_{\mathfrak{p}'}, k(\mathfrak{p}'), k(\mathfrak{p}')) = 0$$

by using [$\mathbf{A_2}$, Lemma 27.7].

So, from $B \to A/\mathfrak{p} \to K$ and the homology sequence

$$H_{n+1}(B, K, K) \to H_{n+1}(A/\mathfrak{p}, K, K) \to H_n(B, A/\mathfrak{p}, K)$$

it follows that $H_{n+1}(A/\mathfrak{p}, K, K) = 0$.

Now considering $A \to A/\mathfrak{p} \to K$ and the homology sequence

$$H_{n+1}(A/\mathfrak{p}, K, K) \to H_n(A, A/\mathfrak{p}, K) \to H_n(A, K, K)$$
$$\to H_n(A/\mathfrak{p}, K, K) \to H_{n-1}(A, A/\mathfrak{p}, K),$$

since the first and the last terms are zero, we conclude the proof.

To get a generalization of Proposition 4.1 we define (locally) a simplicial dimension for any ring B, namely

$$\operatorname{s.dim} B \leqslant r \Leftrightarrow H_i(B_{\mathfrak{p}}, k(\mathfrak{p}), k(\mathfrak{p})) = 0 \quad \text{for all } i > r, \mathfrak{p} \in \operatorname{spec}(B).$$

We want to remark that the above definition coincides with André's when, for B, H_n is equivalent to strongly H_n.

PROPOSITION 4.2. *Let B be any ring, with $\operatorname{s.dim} B = r$, $\mathfrak{b} \subseteq B$ any ideal and $A = B/\mathfrak{b}$. Then for any $\mathfrak{p} \in \operatorname{spec}(A)$ and $n > r$ there is an open set U, $\mathfrak{p} \in U$, such that for every $\mathfrak{q} \in U \cap V(\mathfrak{p})$,* (2) *holds.*

PROOF. By Proposition 3.4, for $\mathfrak{p} \in \operatorname{spec}(A)$ and $n > r$, $U_{H_{n-1}}(A)$ and $U_{H_n}(A/\mathfrak{p})$ are open, so the proof runs as in Proposition 4.1.

COROLLARY 4.3. *Under the hypothesis of Proposition* 4.1 (*or Proposition* 4.2) *for all $\mathfrak{p} \in \operatorname{spec}(A)$ we can find a neighbourhood of $\mathfrak{p}$, U, such that δ_n is constant in $U \cap V(\mathfrak{p})$, for $n > 3$ (resp. $n > r$).*

PROOF. By Proposition 4.1 (or Proposition 4.2) there exists a neighbourhood of $\mathfrak{p}$ in which (2) holds, but by assumption on A, $U_{H_n}(A)$ is open, so we can get a neighbourhood of $\mathfrak{p}$ such that for any $\mathfrak{q}$, $\delta_n(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) = 0$. Now the corollary follows.

Since a C.I. ring A is characterized by $\delta_3(A) = 0$, the first class of rings that are not C.I., in this direction, is characterized by $\delta_3(A) = 1$. The property for a local ring A such that $\delta_3(A) = 1$ will be denoted briefly $\delta_3(1)$.

PROPOSITION 4.4. *Let A be an excellent ring such that $A_{\mathfrak{p}}$ is not C.I. for all $\mathfrak{p} \in \operatorname{spec}(A)$, then $U_{\delta_3(1)}(A)$ is open.*

PROOF. For such a ring the property $\delta_3(1)$ is stable under generalizations by Theorem 1.7 and the hypothesis on the localizations. So it is enough to check that for every $\mathfrak{p} \in U_{\delta_3(1)}(A)$, $V(\mathfrak{p}) \cap U_{\delta_3(1)}(A)$ contains a nonempty open subset.

Take $\mathfrak{p} \in U_{\delta_3(1)}(A)$; as in Proposition 3.6 we can shrink to an open neighbourhood U of $\mathfrak{p}$ in which for any $\mathfrak{q} \in U \cap V(\mathfrak{p})$ we have $\delta_3(A_\mathfrak{q}) \leq \delta_3(A_\mathfrak{p})$, but $\delta_3(A_\mathfrak{q}) = 0$ implies $\delta_3(A_\mathfrak{p}) = 0$, so $\delta_3(A_\mathfrak{q}) = 1$.

Now we get a proposition that we can think of as a kind of Nagata criterion for "δ_n upper-semicontinuous".

THEOREM 4.5. *Let A be any ring, n a fixed integer and s any integer such that $\Delta_n(s)$ is stable under generalizations on A. If for every $\mathfrak{p} \in U_{\Delta_n(s)}(A)$, let us say $\delta_n(A_\mathfrak{p}) = r$, $U_{\Delta_n(s-r)}(A/\mathfrak{p})$ contains a nonempty open subset, then $U_{\Delta_n(s)}(A)$ is open, i.e. δ_n is upper-semicontinuous on A.*

PROOF. Take $\mathfrak{p} \in U_{\Delta_n(s)}(A)$; with $\delta_n(A) = r$, we can find a neighbourhood of $\mathfrak{p}$ in $V(\mathfrak{p})$ such that $\operatorname{spec}(A/\mathfrak{p}) = U_{\Delta_n(s-r)}(A/\mathfrak{p})$; now by Theorem 2.3 there is a neighbourhood in which for any $\mathfrak{q}$

$$\delta_n(A_\mathfrak{q}) \leq \delta_n(A_\mathfrak{p}) + \delta_n(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$$

so in the common neighbourhood we have

$$\delta_n(A_\mathfrak{q}) \leq r + s - r = s;$$

this concludes the proof.

REMARK 4.6. Of course the first hypothesis in the previous proposition holds for $n \leq 3$ or, if A contains a field of characteristic 0, for all n even. The second hypothesis holds if for every $\mathfrak{p} \in U_{\Delta_n(s)}(A)$, $U_{H_n}(A/\mathfrak{p})$ contains a nonempty open subset. The criterion also holds under the assumption that for every $\mathfrak{p} \in \operatorname{spec}(A)$, $U_{\Delta_n(s)}(A/\mathfrak{p})$ contains a nonempty open subset.

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