DIFFERENTIAL ALGEBRAIC GROUP STRUCTURES ON THE PLANE

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Abstract. The differential algebraic group structures on the affine line and plane are classified. The additive group $G_a$ of the coefficient field is the only differential algebraic group structure on the line. Every differential algebraic group with underlying set in the plane is unipotent and is isomorphic to a group whose law of composition is defined by the formula

$$(u_1, u_2)(v_1, v_2) = (u_1 + v_1, u_2 + v_2 + f(u_1, v_1)),$$

where $f$ is a 2-cocycle of $G_a$ into $G_a$.

Throughout, $\mathcal{U}$ will be a fixed universal differential field of characteristic 0, equipped with a finite set $\Delta$ of derivation operators that commute with each other. $\mathcal{K}$ will denote the field of constants of $\mathcal{U}$.

A differential algebraic group is, roughly speaking, a group object in the category of differential algebraic sets. The most concrete example, studied in [1], has as its underlying set a differential variety in the sense of Kolchin and Ritt. Thus, it is the solution set in affine space $A^n$ of finitely many differential polynomial equations with coefficients in $\mathcal{K}$.

We refer to the well-known algebraic groups by the usual symbols. The subgroup of the general linear group $GL(n)$ consisting of all upper triangular matrices will be denoted by $T(n)$, the subgroup of $T(n)$ consisting of all upper triangular unipotent matrices is denoted by $T(n, 1)$, and the additive group of the $\mathcal{U}$-vector space $\mathcal{U}^n$ is denoted by $G_a^n$.

We shall use the prefix "$\Delta$-" or "$\delta$-" if $\Delta = \{\delta\}$, in place of "differential algebraic" or "differential rational". Thus, we shall speak of "$\Delta$-groups", "$\Delta$-maps", and "$\Delta$-functions on $G$".

We denote the differential algebra over $\mathcal{U}$ (resp. differential field extension of $\mathcal{U}$) generated by $t_1, \ldots, t_d$ by $\mathcal{U}\{t_1, \ldots, t_d\}$ (resp. by $\mathcal{U}\langle t_1, \ldots, t_d\rangle$). If $G$ is a $\Delta$-group, the differential field of $\Delta$-functions on $G$ is denoted by $\mathcal{U}\langle G\rangle$.

A $\Delta$-group $G$ is unipotent if $G$ has a normal sequence of $\Delta$-subgroups whose successive quotients are $\Delta$-isomorphic to $\Delta$-subgroups of $G_a$. If the cardinality of $\Delta$ is 1, any such normal sequence can be refined so that successive quotients are $\Delta$-isomorphic to $G_a$ or to the additive group $(G_a)_\mathcal{K}$ of the field $\mathcal{K}$ of constants of $\mathcal{U}$.

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The purpose of this note is to establish the following result, which describes the \( \delta \)-group structures that can be defined on affine spaces of low dimension.

**Theorem.** Let \( G \) be a \( \delta \)-group whose underlying \( \delta \)-set is \( \delta \)-isomorphic to \( A^d \), where \( d \) is \( < 2 \). Then \( G \) is unipotent. If \( d = 1 \), \( G \) is \( \delta \)-isomorphic to \( G_a \). If \( d = 2 \), there is a 2-cocycle \( f \) of \( G_a \) into \( G_a \) of the form \( f(u, v) = \Sigma_{1 < j \leq d} a_{ij} u^i v^j \) such that \( G \) is \( \delta \)-isomorphic to the group whose underlying \( \delta \)-set is \( A^2 \) and whose law of composition is given by the formula

\[
(u_1, u_2)(v_1, v_2) = (u_1 + v_1, u_2 + v_2 + f(u_1, v_1)).
\]

As contrasted with the situation for algebraic sets, it is not always the case that the differential algebra \( \mathcal{O} \) over \( \mathbb{R} \) of everywhere defined \( \mathbb{A} \)-functions on an affine differential algebraic set is finitely generated, since it need not be equal to the differential coordinate ring. However, if \( G \) is a \( \Delta \)-group whose underlying \( \Delta \)-set is \( \Delta \)-isomorphic to \( A^d \) then there are \( d \) elements \( t_1, \ldots, t_d \), differentially algebraically independent over \( \mathcal{O} \), such that \( \Theta = \mathcal{O} \{ t_1, \ldots, t_d \} \), i.e., \( \Theta \) is a finitely-generated differential polynomial algebra over \( \mathcal{O} \) (this follows from the fact that \( \Theta \) is differentially isomorphic over \( \mathcal{O} \) to the differential algebra over \( \mathcal{O} \) of everywhere defined \( \Delta \)-functions on \( A^d \), which is equal to the differential coordinate ring).

**Proposition 1.** Let \( G \) be a \( \Delta \)-subgroup of \( \text{GL}(n) \) whose underlying \( \Delta \)-set is \( \Delta \)-isomorphic to affine \( d \)-space \( A^d \). The additive group of the Lie algebra \( l(G) \) of matrices of \( G \) is \( \Delta \)-isomorphic to \( G_a^d \).

**Proof.** In [1] we define the tangent space to \( G \) at the identity element 1 as follows. A \( \mathcal{O} \)-linear map \( T \) from the local differential ring of \( \Delta \)-functions on \( G \) defined at 1 is called a differential tangent vector if \( T(fg) = T(f)g(1) + f(1)T(g) \) and \( \delta(T(f)) = T(\delta f) \) \((f, g \ \Delta \)-functions on \( G \) defined at 1, \( \delta \in \Delta \)). The tangent space at 1 has a structure of additive group defined on it in the obvious way. This group is isomorphic to the additive group of the Lie algebra \( l(G) \) of matrices of \( G \). This isomorphism defines on the tangent space to \( G \) at the identity element a structure of differential algebraic group.

In [4], Kolchin shows that if \( \iota = (t_1, \ldots, t_d) \) is any family of generators of the differential field of \( \Delta \)-functions on \( G \), with \( t_1, \ldots, t_d \) defined at 1, then the tangent space to \( G \) at 1, hence the additive group of \( l(G) \), is \( \Delta \)-isomorphic to a \( \Delta \)-subgroup \( V \) of \( G_a^d \) constructed as follows. Let \( p \) be the defining differential ideal of \( \iota \) over \( \mathcal{O} \) in the differential polynomial algebra \( \mathcal{O} \{ y_1, \ldots, y_d \} \) and let \( p_1 \) be the differential ideal in \( \mathcal{O} \{ y_1, \ldots, y_d \} \) generated by the homogeneous linear differential polynomials

\[
P_1 = \sum_{1 < j < d, \theta \in \Theta} \frac{\partial P}{\partial y_j} (t(1)) \theta y_j \quad \text{with } P \in p.
\]

\( V \) is the set of zeros in \( A^d \) of \( p_1 \). In our case, since the underlying \( \Delta \)-set of \( G \) is \( \Delta \)-isomorphic to \( A^d \), we can find everywhere defined \( \Delta \)-functions \( t_1, \ldots, t_d \) on \( G \), which are differentially algebraically independent over \( \mathcal{O} \), such that \( \mathcal{O} \langle G \rangle = \mathcal{O} \langle t_1, \ldots, t_d \rangle \). It follows that \( p \) and \( p_1 \) are equal to the zero ideal, whence \( V = G_a^d \).
We now prove a weak analog of a theorem of M. Lazard [5] about algebraic group structures on affine space. It is easy to see that if a $\Delta$-subgroup of GL($n$) is connected and solvable, then its Zariski closure, which is an algebraic subgroup of GL($n$), also is connected and solvable; it follows that a connected $\Delta$-subgroup of GL($n$) is solvable if and only if it is conjugate in GL($n$) to a subgroup of T($n$).

**PROPOSITION 2.** A solvable $\Delta$-subgroup $G$ of GL($n$) whose underlying $\Delta$-set is $\Delta$-isomorphic to $A^d$ is unipotent.

**Proof.** We may assume that $G \subset T(n)$. Since the underlying $\Delta$-set of $G$ is $\Delta$-isomorphic to $A^d$, the differential algebra over $\mathbb{Q}$ of everywhere defined $\Delta$-functions on $G$ is a finitely-generated differential polynomial algebra. It follows that if $f$ and $1/f$ are everywhere defined $\Delta$-functions on $G$ then $f \in \mathbb{Q}_l$, i.e., is constant on $G$. Let $f_i$ be the everywhere defined $\Delta$-function on $G$ whose value at $a = (a_j)$ is $a_i$. Since $G$ is upper triangular, $f_i$ and $1/f_i$ are both everywhere defined on $G$. Therefore, $f_i$ is constant on $G$. Since $f_i(1) = 1$, it follows that $G$ is upper triangular unipotent.

In [3] a Lie algebra $g$ is defined to be differential algebraic if the additive group of $g$ is a $\Delta$-group and the Lie product and scalar multiplication maps are everywhere defined $\Delta$-maps. The Lie algebra $gl(n)$ of $n \times n$ matrices with entries in $\mathbb{Q}$ is clearly differential algebraic. A $\mathbb{K}$-subalgebra $g$ of $gl(n)$ is a $\Delta$-subalgebra if and only if $g$ is the set of zeros of a homogeneous linear differential ideal. In particular, the Lie algebra of matrices of a $\Delta$-subgroup of GL($n$) is differential algebraic.

In [3] a differential algebraic Lie algebra $g$ is said to be solvable (resp. nilpotent) if $g$ is solvable (resp. nilpotent) as a Lie algebra over $\mathbb{K}$. Thus, there must exist a sequence, $g = g^0 \supset g^1 \supset \cdots \supset g^r = 0$, of ideals of $g$ such that $g^i/g^{i+1}$ is abelian (resp. central in $g/g^{i+1}$), $0 < i < r - 1$. A $\Delta$-subalgebra $g$ of $gl(n)$ is solvable if and only if there is a matrix $g \in GL(n)$ such that every matrix in $gg^{-1}$ is upper triangular [3, Proposition 11]. $g$ consists of nilpotent matrices if and only if there is a matrix $g \in GL(n)$ such that every matrix in $gg^{-1}$ is upper triangular nilpotent. In this case, $g$ is nilpotent [3, Proposition 10].

**PROPOSITION 3.** A connected $\Delta$-subgroup $G$ of GL($n$) is solvable if and only if $l(G)$ is solvable. $G$ is unipotent if and only if $l(G)$ consists of nilpotent matrices.

**Proof.** $G$ is solvable if and only if there is a matrix $g \in GL(n)$ such that $gg^{-1} \subset T(n)$. Therefore, $G$ is solvable if and only if there is a matrix $g \in GL(n)$ such that $l(gg^{-1})$ is upper triangular. Since $l(gg^{-1}) = gl(G)g^{-1}$ [1, p. 929], $G$ is solvable if and only if $l(G)$ is solvable.

$G$ is unipotent if and only if there is a matrix $g \in GL(n)$ such that $gg^{-1} \subset T(n, 1)$ [2, p. 89].

Therefore, $G$ is unipotent if and only if there is a matrix $g \in GL(n)$ such that $l(gg^{-1})$ consists of upper triangular matrices with 0's on the diagonal. Thus, $G$ is unipotent if and only if $l(G)$ consists of nilpotent matrices.

**COROLLARY.** Let $G$ be a solvable $\Delta$-subgroup of GL($n$) whose underlying $\Delta$-set is $\Delta$-isomorphic to $A^d$. Then $l(G)$ consists of nilpotent matrices.
We turn now to a proof of the theorem.

Let $G$ be a $\delta$-group whose underlying $\delta$-set is $\delta$-isomorphic to $A^d$. Since the algebra of everywhere defined $\delta$-functions on $G$ is a finitely-generated differential polynomial algebra over $\mathbb{Q}$, $G$ is linear [1, p. 914]. Therefore, we may assume that $G$ is a $\delta$-subgroup of $GL(n)$ for some $n$.

Suppose the differential dimension of $G$ is 1. By Proposition 1, the additive group of $l(G)$ is $\delta$-isomorphic to $G_a$. There are two isomorphism classes of $\delta$-Lie algebras whose additive groups are equal to $G_a$. One is represented by the abelian Lie algebra $g_a$, and the other by the substitution Lie algebra $q_s$, where the Lie product is given by the formula $[u, v] = u \cdot v - v \cdot u$ [3, Theorem 5]. $q_s$ is not $\delta$-isomorphic to the Lie algebra of matrices of a differential algebraic matrix group [3, Theorem 6]. Therefore, $l(G)$ is $\delta$-isomorphic to $g_a$. In particular, $G$ is commutative [1, p. 928], hence is well known to be simultaneously triangularizable. Thus, $G$ is unipotent by Proposition 2. Since $G$ is a commutative unipotent $\delta$-group whose underlying $\delta$-set is $\delta$-isomorphic to $A^1$, $G$ is $\delta$-isomorphic to $G_a$ [2, p. 95].

Suppose the differential dimension of $G$ is 2. There are thirteen general types of $\delta$-Lie algebras whose additive groups are equal to $G_a \times G_a$ [3, §2]. Ten of these are of substitutional type and three are of finite type.

A $\delta$-Lie algebra $g$ whose additive group is $G_a \times G_a$ is $\delta$-isomorphic to the Lie algebra of matrices of a differential algebraic matrix group if and only if $g$ is of finite type [3, Theorem 8]. $g$ is solvable if and only if $g$ is of finite type [3, Theorem 7]. Therefore, $l(G)$ is solvable and is $\delta$-isomorphic to a $\delta$-Lie algebra of finite type. Since $l(G)$ is solvable, $G$ is solvable by Proposition 3. Since the underlying $\delta$-set of $G$ is $\delta$-isomorphic to $A^2$, $G$ is unipotent by Proposition 2, and $l(G)$ consists of nilpotent matrices, and is nilpotent. There is a short exact sequence of $\delta$-Lie algebras and homomorphisms of $\delta$-Lie algebras such that $\iota'(q_a)$ is central in $l(G)$ [3, discussion following Proposition 13]. As in the case of algebraic groups of characteristic 0, the Lie algebra of matrices of the center $Z$ of $G$ is the center $Z$ of $l(G)$ [3, Corollary 2 of Proposition 16]. Since $Z$ is commutative and unipotent, the map $\log: Z \to Z$ and its inverse $\exp: Z \to Z$ are $\delta$-homomorphisms. Let $\iota = \exp \circ \iota'. \iota$ is a $\delta$-homomorphism from $g_a = G_a$ onto a central $\delta$-subgroup of $G$. Moreover, $l(\iota(G_a)) = \iota'(g_a)$. Let $G'$ be a differential algebraic matrix group $\delta$-isomorphic to $G/\iota(G_a)$. Then $l(G')$ is $\delta$-isomorphic to $l(G)/\iota'(g_a)$ [1, Propositions 22 and 29]. Therefore, $l(G')$ is $\delta$-isomorphic to $g_a$. Since $G$ is unipotent, $G'$ is unipotent, whence, as above, $G'$ is $\delta$-isomorphic to $G_a$. Thus, $G$ is a central extension of $G_a$ by $G_a$. Every central extension of $G_a$ by $G_a$ is $\delta$-isomorphic to a central extension whose underlying $\delta$-set is $A^2$ and whose law of composition is given by the formula

$$(u_1, u_2)(v_1, v_2) = (u_1 + v_1, u_2 + v_2 + f(u_1, v_1)), $$

where $f$ is the $\delta$-2-cocycle of $G_a$ in $G_a$ whose value at $(u, v)$ is $\Sigma_{i<j} a_{ij} u^{i} v^{j}$ by [2, corollary to Theorem 7].
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