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DIFFERENTIAL ALGEBRAIC GROUP STRUCTURES ON THE PLANE

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ABSTRACT. The differential algebraic group structures on the affine line and plane are classified. The additive group G_a of the coefficient field is the only differential algebraic group structure on the line. Every differential algebraic group with underlying set in the plane is unipotent and is isomorphic to a group whose law of composition is defined by the formula

$$(u_1, u_2)(v_1, v_2) = (u_1 + v_1, u_2 + v_2 + f(u_1, v_1)),$$

where f is a 2-cocycle of G_a into G_a.

Throughout, $\mathfrak{U}$ will be a fixed universal differential field of characteristic 0, equipped with a finite set Δ of derivation operators that commute with each other. $\mathfrak{K}$ will denote the field of constants of $\mathfrak{U}$.

A differential algebraic group is, roughly speaking, a group object in the category of differential algebraic sets. The most concrete example, studied in [1], has as its underlying set a differential variety in the sense of Kolchin and Ritt. Thus, it is the solution set in affine space A^n of finitely many differential polynomial equations with coefficients in $\mathfrak{U}$.

We refer to the well-known algebraic groups by the usual symbols. The subgroup of the general linear group $\mathrm{GL}(n)$ consisting of all upper triangular matrices will be denoted by $T(n)$, the subgroup of $T(n)$ consisting of all upper triangular unipotent matrices is denoted by $T(n, 1)$, and the additive group of the $\mathfrak{U}$-vector space $\mathfrak{U}^n$ is denoted by G_a^n.

We shall use the prefix "Δ-", or "δ-" if $\Delta = \{\delta\}$, in place of "differential algebraic" or "differential rational". Thus, we shall speak of "Δ-groups", "Δ-maps", and "Δ-functions on G".

We denote the differential algebra over $\mathfrak{U}$ (resp. differential field extension of $\mathfrak{U}$) generated by $t_1, \ldots, t_d$ by $\mathfrak{U}\{t_1, \ldots, t_d\}$ (resp. by $\mathfrak{U}\langle t_1, \ldots, t_d\rangle$). If G is a Δ-group, the differential field of Δ-functions on G is denoted by $\mathfrak{U}\langle G\rangle$.

A Δ-group G is *unipotent* if G has a normal sequence of Δ-subgroups whose successive quotients are Δ-isomorphic to Δ-subgroups of G_a. If the cardinality of Δ is 1, any such normal sequence can be refined so that successive quotients are Δ-isomorphic to G_a or to the additive group $(G_a)_{\mathfrak{K}}$ of the field $\mathfrak{K}$ of constants of $\mathfrak{U}$.

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The purpose of this note is to establish the following result, which describes the δ-group structures that can be defined on affine spaces of low dimension.

THEOREM. *Let G be a δ-group whose underlying δ-set is δ-isomorphic to A^d, where d is ≤ 2. Then G is unipotent. If $d = 1$, G is δ-isomorphic to G_a. If $d = 2$, there is a 2-cocycle f of G_a into G_a of the form $f(u, v) = \sum_{i<j} a_{ij} u^{(i)} v^{(j)}$ such that G is δ-isomorphic to the group whose underlying δ-set is A^2 and whose law of composition is given by the formula*

$$(u_1, u_2)(v_1, v_2) = (u_1 + v_1, u_2 + v_2 + f(u_1, v_1)).$$

As contrasted with the situation for algebraic sets, it is not always the case that the differential algebra $\mathcal{O}$ over $\mathfrak{U}$ of everywhere defined Δ-functions on an affine differential algebraic set is finitely generated, since it need not be equal to the differential coordinate ring. However, if G is a Δ-group whose underlying Δ-set is Δ-isomorphic to A^d then there are d elements $t_1, \ldots, t_d$, differentially algebraically independent over $\mathfrak{U}$, such that $\mathcal{O} = \mathfrak{U}\{t_1, \ldots, t_d\}$, i.e., $\mathcal{O}$ is a finitely-generated differential polynomial algebra over $\mathfrak{U}$ (this follows from the fact that $\mathcal{O}$ is differentially isomorphic over $\mathfrak{U}$ to the differential algebra over $\mathfrak{U}$ of everywhere defined Δ-functions on A^d, which is equal to the differential coordinate ring).

PROPOSITION 1. *Let G be a Δ-subgroup of* GL(n) *whose underlying Δ-set is Δ-isomorphic to affine d-space A^d. The additive group of the Lie algebra $l(G)$ of matrices of G is Δ-isomorphic to G_a^d.*

PROOF. In [1] we define the tangent space to G at the identity element 1 as follows. A $\mathfrak{U}$-linear map T from the local differential ring of Δ-functions on G defined at 1 is called a differential tangent vector if $T(fg) = T(f)g(1) + f(1)T(g)$ and $\delta(T(f)) = T(\delta f)$ (f, g Δ-functions on G defined at $1, \delta \in \Delta$). The tangent space at 1 has a structure of additive group defined on it in the obvious way. This group is isomorphic to the additive group of the Lie algebra $l(G)$ of matrices of G. This isomorphism defines on the tangent space to G at the identity element a structure of differential algebraic group.

In [4], Kolchin shows that if $t = (t_1, \ldots, t_d)$ is any family of generators of the differential field of Δ-functions on G, with $t_1, \ldots, t_d$ defined at 1, then the tangent space to G at 1, hence the additive group of $l(G)$, is Δ-isomorphic to a Δ-subgroup V of G_a^d constructed as follows. Let p be the defining differential ideal of t over $\mathfrak{U}$ in the differential polynomial algebra $\mathfrak{U}\{y_1, \ldots, y_d\}$ and let p_1 be the differential ideal in $\mathfrak{U}\{y_1, \ldots, y_d\}$ generated by the homogeneous linear differential polynomials

$$P_1 = \sum_{1 \leq j \leq d, \theta \in \Theta} \frac{\partial P}{\partial \theta y_j}(t(1))\theta y_j \quad \text{with } P \in p.$$

V is the set of zeros in A^d of p_1. In our case, since the underlying Δ-set of G is Δ-isomorphic to A^d, we can find everywhere defined Δ-functions $t_1, \ldots, t_d$ on G, which are differentially algebraically independent over $\mathfrak{U}$, such that $\mathfrak{U}\langle G\rangle = \mathfrak{U}\langle t_1, \ldots, t_d\rangle$. It follows that p and p_1 are equal to the zero ideal, whence $V = G_a^d$.

We now prove a weak analog of a theorem of M. Lazard [5] about algebraic group structures on affine space. It is easy to see that if a Δ-subgroup of $\mathrm{GL}(n)$ is connected and solvable, then its Zariski closure, which is an algebraic subgroup of $\mathrm{GL}(n)$, also is connected and solvable; it follows that a connected Δ-subgroup of $\mathrm{GL}(n)$ is solvable if and only if it is conjugate in $\mathrm{GL}(n)$ to a subgroup of $T(n)$.

Proposition 2. *A solvable Δ-subgroup G of* $\mathrm{GL}(n)$ *whose underlying Δ-set is Δ-isomorphic to A^d is unipotent.*

Proof. We may assume that $G \subset T(n)$. Since the underlying Δ-set of G is Δ-isomorphic to A^d, the differential algebra over $\mathfrak{U}$ of everywhere defined Δ-functions on G is a finitely-generated differential polynomial algebra. It follows that if f and $1/f$ are everywhere defined Δ-functions on G then $f \in \mathfrak{U}$, i.e., is constant on G. Let f_i be the everywhere defined Δ-function on G whose value at $a = (a_{ij})$ is a_{ii}. Since G is upper triangular, f_i and $1/f_i$ are both everywhere defined on G. Therefore, f_i is constant on G. Since $f_i(1) = 1$, it follows that G is upper triangular unipotent.

In [3] a Lie algebra $\mathfrak{g}$ is defined to be differential algebraic if the additive group of $\mathfrak{g}$ is a Δ-group and the Lie product and scalar multiplication maps are everywhere defined Δ-maps. The Lie algebra $\mathfrak{g}l(n)$ of $n \times n$ matrices with entries in $\mathfrak{U}$ is clearly differential algebraic. A $\mathfrak{K}$-subalgebra $\mathfrak{g}$ of $\mathfrak{g}l(n)$ is a Δ-subalgebra if and only if $\mathfrak{g}$ is the set of zeros of a homogeneous linear differential ideal. In particular, the Lie algebra of matrices of a Δ-subgroup of $\mathrm{GL}(n)$ is differential algebraic.

In [3] a differential algebraic Lie algebra $\mathfrak{g}$ is said to be solvable (resp. nilpotent) if $\mathfrak{g}$ is solvable (resp. nilpotent) as a Lie algebra over $\mathfrak{K}$. Thus, there must exist a sequence, $\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^r = 0$, of ideals of $\mathfrak{g}$ such that $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ is abelian (resp. central in $\mathfrak{g}/\mathfrak{g}^{i+1}$), $0 \leqslant i \leqslant r-1$. A Δ-subalgebra $\mathfrak{g}$ of $\mathfrak{g}l(n)$ is solvable if and only if there is a matrix $g \in \mathrm{GL}(n)$ such that every matrix in $g\mathfrak{g}g^{-1}$ is upper triangular [3, Proposition 11]. $\mathfrak{g}$ consists of nilpotent matrices if and only if there is a matrix $g \in \mathrm{GL}(n)$ such that every matrix in $g\mathfrak{g}g^{-1}$ is upper triangular nilpotent. In this case, $\mathfrak{g}$ is nilpotent [3, Proposition 10].

Proposition 3. *A connected Δ-subgroup G of* $\mathrm{GL}(n)$ *is solvable if and only if $l(G)$ is solvable. G is unipotent if and only if $l(G)$ consists of nilpotent matrices.*

Proof. G is solvable if and only if there is a matrix $g \in \mathrm{GL}(n)$ such that $gGg^{-1} \subset T(n)$. Therefore, G is solvable if and only if there is a matrix $g \in \mathrm{GL}(n)$ such that $l(gGg^{-1})$ is upper triangular. Since $l(gGg^{-1}) = gl(G)g^{-1}$ [1, p. 929], G is solvable if and only if $l(G)$ is solvable.

G is unipotent if and only if there is a matrix $g \in \mathrm{GL}(n)$ such that $gGg^{-1} \subset T(n, 1)$ [2, p. 89].

Therefore, G is unipotent if and only if there is a matrix $g \in \mathrm{GL}(n)$ such that $l(gGg^{-1})$ consists of upper triangular matrices with 0's on the diagonal. Thus, G is unipotent if and only if $l(G)$ consists of nilpotent matrices.

Corollary. *Let G be a solvable Δ-subgroup of* $\mathrm{GL}(n)$ *whose underlying Δ-set is Δ-isomorphic to A^d. Then $l(G)$ consists of nilpotent matrices.*

We turn now to a proof of the theorem.

Let G be a δ-group whose underlying δ-set is δ-isomorphic to A^d. Since the algebra of everywhere defined δ-functions on G is a finitely-generated differential polynomial algebra over $\mathfrak{U}$, G is linear [**1**, p. 914]. Therefore, we may assume that G is a δ-subgroup of $\mathrm{GL}(n)$ for some n.

Suppose the differential dimension of G is 1. By Proposition 1, the additive group of $l(G)$ is δ-isomorphic to G_a. There are two isomorphism classes of δ-Lie algebras whose additve groups are equal to G_a. One is represented by the abelian Lie algebra $\mathfrak{g}_a$ and the other by the substitution Lie algebra $\mathfrak{g}_s$, where the Lie product is given by the formula $[u, v] = u \cdot \delta v - \delta u \cdot v$ [**3**, Theorem 5]. $\mathfrak{g}_s$ is not δ-isomorphic to the Lie algebra of matrices of a differential algebraic matrix group [**3**, Theorem 6]. Therefore, $l(G)$ is δ-isomorphic to $\mathfrak{g}_a$. In particular, G is commutative [**1**, p. 928], hence is well known to be simultaneously triangularizable. Thus, G is unipotent by Proposition 2. Since G is a commutative unipotent δ-group whose underlying δ-set is δ-isomorphic to A^1, G is δ-isomorphic to G_a [**2**, p. 95].

Suppose the differential dimension of G is 2. There are thirteen general types of δ-Lie algebras whose additive groups are equal to $G_a \times G_a$ [**3**, §2]. Ten of these are of substitutional type and three are of finite type.

A δ-Lie algebra $\mathfrak{g}$ whose additive group is $G_a \times G_a$ is δ-isomorphic to the Lie algebra of matrices of a differential algebraic matrix group if and only if $\mathfrak{g}$ is of finite type [**3**, Theorem 8]. $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}$ is of finite type [**3**, Theorem 7]. Therefore, $l(G)$ is solvable and is δ-isomorphic to a δ-Lie algebra of finite type. Since $l(G)$ is solvable, G is solvable by Proposition 3. Since the underlying δ-set of G is δ-isomorphic to A^2, G is unipotent by Proposition 2, and $l(G)$ consists of nilpotent matrices, and is nilpotent. There is a short exact sequence

$$0 \to \mathfrak{g}_a \xrightarrow{\iota'} l(G) \xrightarrow{\pi'} \mathfrak{g}_a \to 0$$

of δ-Lie algebras and homomorphisms of δ-Lie algebras such that $\iota'(\mathfrak{g}_a)$ is central in $l(G)$ [**3**, discussion following Proposition 13]. As in the case of algebraic groups of characteristic 0, the Lie algebra of matrices of the center Z of G is the center $\mathfrak{Z}$ of $l(G)$ [**3**, Corollary 2 of Proposition 16]. Since Z is commutative and unipotent, the map log: $Z \to \mathfrak{Z}$ and its inverse exp: $\mathfrak{Z} \to Z$ are δ-homomorphisms. Let $\iota = \exp \circ \iota'$. ι is a δ-homomorphism from $\mathfrak{g}_a = G_a$ onto a central δ-subgroup of G. Moreover, $l(\iota(G_a)) = \iota'(\mathfrak{g}_a)$. Let G' be a differential algebraic matrix group δ-isomorphic to $G/\iota(G_a)$. Then $l(G')$ is δ-isomorphic to $l(G)/\iota'(\mathfrak{g}_a)$ [**1**, Propositions 22 and 29]. Therefore, $l(G')$ is δ-isomorphic to $\mathfrak{g}_a$. Since G is unipotent, G' is unipotent, whence, as above, G' is δ-isomorphic to G_a. Thus, G is a central extension of G_a by G_a. Every central extension of G_a by G_a is δ-isomorphic to a central extension whose underlying δ-set is A^2 and whose law of composition is given by the formula

$$(u_1, u_2)(v_1, v_2) = (u_1 + v_1, u_2 + v_2 + f(u_1, v_1)),$$

where f is the δ-2-cocyle of G_a in G_a whose value at (u, v) is $\sum_{i<j} a_{ij} u^{(i)} v^{(j)}$ by [**2**, corollary to Theorem 7].

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