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MORITA EQUIVALENT SEMIGROUPS OF QUOTIENTS

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ABSTRACT. Let $S$ be a monoid and $SM$ and $SN$ be retracts of each other. We show that $\text{End}_S(M)$ and $\text{End}_S(N)$ are Morita equivalent. Using this result, we show that if $A$ and $B$ are Morita equivalent monoids, then their semigroups of quotients are Morita equivalent.

Let $A$ and $B$ be monoids with $0$ and let $A\mathfrak{M}$ (resp. $B\mathfrak{M}$) denote the category of unitary left $A$-systems (resp. $B$-systems). The monoids $A$ and $B$ are Morita equivalent if there are covariant functors $F: A\mathfrak{M} \rightarrow B\mathfrak{M}$ and $G: A\mathfrak{M} \rightarrow B\mathfrak{M}$ with $F \cdot G \simeq 1_{A\mathfrak{M}}$ and $G \cdot F \simeq 1_{B\mathfrak{M}}$. Knauer [3] (and independently, Banaschewski [1]) has shown that when $A$ and $B$ are Morita equivalent, there is an indecomposable projective generator $Q$ in $B\mathfrak{M}$ with $Q \simeq Bb$ where $b^2 = b \in B$ and $1, 1' \in B$ with $b1 = 1$ and $1'1 = e$, the identity of $B$. Moreover, $A \simeq F(Q)$, $A \simeq Hom_B(Q, Q) \simeq bBb$ and $G(X) = BQA \otimes X$.

In the following section, we show that when $S$ is a monoid with $0$ and $SM, SN \in S\mathfrak{M}$ are retracts of each other, the endomorphism semigroups $\text{End}_S(M)$ and $\text{End}_S(N)$ are Morita equivalent. In §2, we use this result to show that when $A$ and $B$ are Morita equivalent monoids, then the McMorris quotient semigroups [7] $Q(A)$ and $Q(B)$ are Morita equivalent. This result was established in [5] for rings.

1. Retracts and Morita equivalence. Let $S$ be a monoid and $SM$ and $SN$ be left $S$-systems. Set $A = \text{End}_S(M)$ and $B = \text{End}_S(N)$; then $A$ and $B$ operate on the right of $M$ and $N$ respectively and so we have bi-systems $\mathfrak{S}MA, \mathfrak{S}NB, \mathfrak{S}\text{Hom}_S(M, N)A$ and $\mathfrak{S}\text{Hom}_S(N, M)B$.

Now we assume that the $S$-system $N$ is a retract of $M$ by means of the mappings $M \xrightarrow{\alpha} N$ with $\alpha \beta = 1_N$. Define the mapping $\mu: \text{Hom}_S(N, M) \otimes_A \text{Hom}(M, N) \rightarrow B$ by $\mu(f \otimes g) = fg$. Then for $b \in B$, $\mu(ba \otimes \beta) = b$ so $\mu$ is onto. Moreover if $\mu(f \otimes g) = \mu(x \otimes y)$, then $fg = xy$ so $f \otimes g = f \otimes ga\beta = fga \otimes \beta = xy\alpha \otimes \beta = x \otimes y\alpha \beta = x \otimes y$. Thus $\mu$ is an isomorphism of $(B, B)$-systems.

Next define $\psi: \text{Hom}_S(M, N) \rightarrow \text{Hom}_A(\text{Hom}_S(N, M), A)$ by $\psi(f)(g) = fg$ and $\psi': \text{Hom}_S(N, M) \rightarrow \text{Hom}_A(\text{Hom}_S(M, N), A)$ by $\psi'(g)(f) = fg$. Then $\psi$ is an $(A, B)$-homomorphism and $\psi'$ is a $(B, A)$-homomorphism. Using $\psi$ we have a commutative diagram

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Figure 1

where \( \tau(m \otimes f) = mf, i(n)(g) = ng \) and \( \nu(m \otimes h)(g) = m \cdot (h(g)) \) respectively. These are all \((A, B)\)-homomorphisms.

**Theorem.** If \( S N \) is a retract of \( S M \), then:
1. \( \psi \) and \( \psi' \) are isomorphisms.
2. All homomorphisms in Figure 1 are isomorphisms.
3. \( Homs(M, N) \) is a cyclic projective left \( A \)-system and a generator as a right \( B \)-system.
4. \( \text{End}_A(Homs(M, N)) \simeq B \) and \( \text{End}_A(Homs(N, M)) \simeq B \).

**Proof.** (1) The inverse of \( \psi \) is \( \psi^{-1}(t) = t(\alpha)\beta \in Homs(M, N) \).

(2) Define \( k: Homs(A, Homs(M, N)) \rightarrow M \otimes A Homs(M, N) \) by \( k(f) = f(\alpha) \otimes \beta \). Then all composition maps from one term to itself are the identity.

(3) \( A Homs(M, N) \) is a retract of \( A \) by \( f \in Homs(M, N) \mapsto f(\alpha) \in Homs(M, M) \) = \( A \) and \( g \in A \rightarrow Homs(M, M) \mapsto g(\alpha) \in Homs(M, N) \). Likewise, \( Homs(N, N) = B \) is a retract of \( Homs(A, N) \) by \( f \in Homs(M, N) \mapsto \beta f \in Homs(M, N) \) and \( g \in Homs(M, N) \mapsto \beta g \in Homs(M, N) \). Thus, \( Homs(M, N) \) is a cyclic projective left \( A \)-system and a generator as a right \( B \)-system.

(4) Let \( Homs(M, N) = At \) and since \( At \) is a retract of \( A \), we can regard \( at \neq bt \) for \( a \neq b \). Now map \( Homs(At, A) \otimes A At \) to \( Homs(At, At) \) by \( f \otimes bt \mapsto [f, bt] \) where \( [f, bt] = (at)f(\alpha)bt \). This mapping is onto since each \( g: At \rightarrow At \) is determined by its image \( g(t) = a_g \), so that if \( h: A \rightarrow A \) is given by \( h(t) = a_g \), then \( [h, t] = g \). To see that this mapping is injective, let \( [h, bt] = [k, ct] \) so that \( (th)bt = (tk)ct \), or \( (th)b = (tk)c \) or \( t(hb) = (tk)c \) and so \( hb = kc \). Thus \( h \otimes bt = hb \otimes t = kc \otimes t = k \otimes ct \) and so \( Homs(A, Homs(M, N), A) \otimes A Homs(M, N) \) is isomorphic to \( \text{End}_A(Homs(M, N)) \). Define \( F: _B \mathfrak{M} \rightarrow _A \mathfrak{M} \) by \( F(X) = Homs(M, N) \otimes _B X \) and \( G: _A \mathfrak{M} \rightarrow _B \mathfrak{M} \) by \( G(Y) = \text{Hom}_S(M, N) \otimes _A Y. \) Then \( G \cdot F \simeq 1_B \mathfrak{M} \). Thus we have the following:

**Theorem.** Let \( S M \) and \( S N \) be retracts of each other. Then (1) \( Homs(M, N) \) is a cyclic projective generator both as a left \( A \)-system and a right \( B \)-system, and (2) \( F \) and \( G \) are inverse equivalences between \( _A \mathfrak{M} \) and \( _B \mathfrak{M} \).

2. **Semigroups of quotients.** A special left quotient filter \( \Sigma \) of a monoid \( B \) is a nonempty collection of left ideals of \( B \) satisfying:

Q1. \( I, J \in \Sigma, f \in \text{Hom}_B(I, B) \Rightarrow f^{-1}(J) \in \Sigma; \)
Q2. if \( J \in \Sigma, Ia^{-1} = \{ s \in B | sa \in I \} \in \Sigma \) for all \( a \in J \), then \( I \in \Sigma; \) and
Q3. \( B \in \Sigma. \)

For each left \( B \)-system \( M \) we define the torsion congruence \( \tau_M \) by \( mn \sim \tau_M \) if there is \( I \in \Sigma \) with \( bm = bn \) for all \( b \in I \). \( M \) is strongly torsion free if \( \tau_M \) is the identity,
and $M$ is torsion if $\tau_M = \omega$. $M$ is $\Sigma$-injective if whenever $C$ is a subsystem of $D$ and $D/C$ is torsion, then each $B$-homomorphism $f: C \to M$ has an extension $f: D \to M$. Each strongly torsion free $B$-system has a unique (up to isomorphism over $M$) extension $E_\Sigma(M)$ which is the minimal $\Sigma$-injective extension of $M$. $E_\Sigma(M)$ is called the $\Sigma$-injective hull of $M$.

Given a special left quotient filter $\Sigma$, let $\mathcal{B} = \bigcup_{I \in \Sigma} \text{Hom}_B(I, B)$ and let $\theta$ be a congruence on $\mathcal{B}$ given by $f \theta g$ if there is some $J \in \Sigma$ on which $f$ and $g$ are both defined and agree. Then $Q_\Sigma(B) = \mathcal{B} / \theta$ with the operation of functional composition $fg: g^{-1}(D_f) \cap D_g \to B$ forms a monoid. Moreover, $Q_\Sigma(B)$ is the $\Sigma$-injective hull of $B$ if $B$ is strongly torsion free. For other definitions and concepts we refer the reader to [4].

Let $A$ and $B$ be Morita equivalent monoids. We use the following notation to describe this situation.

(1) $F: B \mathcal{R} \to A \mathcal{R}$ and $G: A \mathcal{R} \to B \mathcal{R}$ are inverse category equivalences.
(2) $AP_B$ is an $(A, B)$-bisystem which is a left $A$ and right $B$ progenerator (i.e., cyclic projective generator).
(3) $BQA$ is a $(B, A)$-bisystem which is a left $B$ and right $A$ progenerator.
(4) $F = P \otimes_B$ and $G = Q \otimes_A$.
(5) $A \simeq \text{End}_B(P) \simeq \text{End}_B(Q)$ and $B \simeq \text{End}_A(Q) \simeq \text{End}_A(P)$.
(6) $P_B \simeq \text{Hom}_B(P, B)$ and $A_P \simeq \text{Hom}_A(A, Q)$.

We have over-determined our equivalence. For $A$ and $B$ to be Morita equivalent it is necessary and sufficient that $A \simeq \text{End}_B(P)$ for some progenerator $P_B$ [5]. (Although [5] deals only with the additive case, since $P_B$ is a cyclic projective generator, simple modifications of the existing proofs suffice to justify that $A \simeq \text{End}_B(P)$ suffices for $A$ and $B$ to be Morita equivalent.) Since $Q$ is a progenerator and $B$ is a monoid, $B$ and $Q$ are retracts of each other. Without loss of generality, since $B$ is a monoid, $Q$ is indecomposable [3].

Let $\Sigma$ be a special left quotient filter on $B$ and $\mathcal{F}(B)$ be the class of all $\Sigma$-torsion $B$-systems. $\Sigma$ uniquely determines and is determined by $\mathcal{F}(B)$, where $\mathcal{F}(B)$ is special in the sense that if $f \in \text{Hom}(M, N)$ is $\Sigma$-restricted (i.e., $f^{-1}(0) = \{0\}$) and $N \in \mathcal{F}(B)$ then $M \in \mathcal{F}(B)$ [4]. Let $\mathcal{F}(A) = \{N \in A \mathcal{R} | G(N) \in \mathcal{F}(B)\}$. Since $G$ is an equivalence, $\mathcal{F}(A)$ is closed under quotients, disjoint unions, and extensions and so is a torsion class for $A \mathcal{R}$. Moreover, $\mathcal{F}(A)$ is a special torsion class, for if $f \in \text{Hom}_A(M, N)$ is $\Sigma$-restricted and $N \in \mathcal{F}(A)$, then $G(N) \in \mathcal{F}(B)$ and $G(f): G(M) \to G(N)$. Suppose $G(K) \subset G(M)$ with $G(f)(G(K)) = 0$; then since $G$ is an equivalence, $f(K) = 0$ so $K = 0$ since $f$ is $\Sigma$-restricted. Thus $G(K) = 0$ and $G(f)$ is $\Sigma$-restricted. Since $\mathcal{F}(B)$ is a special torsion class, $G(M) \in \mathcal{F}(B)$ so $M \in \mathcal{F}(A)$. Thus $\mathcal{F}(A)$ is special and determines a unique special left quotient filter $\Gamma$ on $A$ by $\Gamma = \{J | G(A/J) \in \mathcal{F}(B)\}$. Thus we have

THEOREM. $\Gamma = \{J | A/J \in \mathcal{F}(A)\}$ is a special left quotient filter on $A$.

Now let $M \in B \mathcal{R}$; then since $F$ and $G$ are inverse equivalences, $M$ is $\Sigma$-injective if and only if $F(M)$ is $\Gamma$-injective. This yields the following

PROPOSITION. $F(E_\Sigma(M)) = E_\Gamma(F(M))$. 

Since \( BQ \) is a finitely generated projective generator, \( BQ \) and \( BB \) are retracts of each other. Knauer [3] (and independently Banaschewski [1]) has shown that \( Q \simeq Bb \) where \( b^2 = b \in B \). Now if \( B \) is strongly torsion free, the isomorphism between \( Q \) and \( Bb \) shows that \( Q \) is strongly torsion free. Then by the definition of \( \Gamma \) and the fact that \( F \) and \( G \) are category equivalences, \( A \) is strongly torsion free.

Now since \( BQ \) and \( BB \) are retracts of each other, \( E^Q(Q) \) and \( E^B(B) \) are retracts of each other. Since \( F(Q) \simeq A \) we have

\[
\text{End}_B(E^Q(Q)) \simeq \text{End}_A(F(E^Q(Q))) \simeq \text{End}_A(E^._1(F(Q))) \simeq \text{End}_A(E^._1(A))
\]

but \( Q^B(B) \simeq \text{End}_B(E^B(B)) \) and \( Q^B(A) \simeq \text{End}_A(E^._1(A)) \simeq \text{End}_B(E^B(Q)) \).

Let \( M = E^Q(Q) \) and \( N = E^B(B) \); then from the last section we see that \( Q^B(B) \) and \( Q^B(A) \) are Morita equivalent and \( \text{Hom}_B(E^Q(Q), E^B(B)) \) is a cyclic projective generator both as a left \( Q(A) \)-system and right \( Q(B) \)-system.

Since \( P \otimes_B E^B(B) \) is \( \Gamma \)-injective and \( E^._1(A) \) is \( A \in \mathbb{E}(A) \),

\[
\text{Hom}_A(A, P \otimes_B E^B(B)) \simeq \frac{\text{Hom}_A(E^._1(A), P \otimes_B E^B(B))}{\text{Hom}_A(E^._1(A)/A, P \otimes_B E^B(B))},
\]

but since \( P \otimes_B E^B(B) \) is strongly torsion free,

\[
\text{Hom}_A(E^._1(A)/A, P \otimes_B E^B(B)) = 0,
\]

and \( \text{Hom}_A(A, P \otimes_B E^B(B)) \) and \( \text{Hom}(E^._1(A), P \otimes_B E^B(B)) \) are isomorphic as right \( Q^B(B) \)-systems. Thus

\[
\text{Hom}_B(E^Q(Q), E^B(B)) \simeq \text{Hom}_A(E^._1(A), P \otimes_B E^B(B)) \simeq \text{Hom}_A(A, P \otimes_B E^B(B)) \simeq P \otimes_B E^B(B) \simeq P \otimes_B Q^B(B).
\]

Thus we have proved the following

**Theorem.** Let \( \Sigma \) be a special left quotient filter on the monoid \( B \) and let \( A \) and \( B \) be Morita equivalent monoids. If \( \Gamma \) is the corresponding special left quotient filter on \( A \), then \( Q^B(A) \) and \( Q^B(B) \) are Morita equivalent. Moreover, in the situation above, \( P \otimes_B Q^B(B) \) is right \( Q^B(B) \)-cyclic projective generator and

\[
Q^B(A) \simeq \text{End}_{Q^B(B)}(P \otimes_B Q^B(B)).
\]

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**Bibliography**