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MORITA EQUIVALENT SEMIGROUPS OF QUOTIENTS

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ABSTRACT. Let S be a monoid and ${}_SM$ and ${}_SN$ be retracts of each other. We show that $\mathrm{End}_S(M)$ and $\mathrm{End}_S(N)$ are Morita equivalent. Using this result, we show that if A and B are Morita equivalent monoids, then their semigroups of quotients are Morita equivalent.

Let A and B be monoids with 0 and let ${}_A\mathfrak{M}$ (resp. ${}_B\mathfrak{M}$) denote the category of unitary left A-systems (resp. B-systems). The monoids A and B are Morita equivalent if there are covariant functors $F\colon {}_B\mathfrak{M} \to {}_A\mathfrak{M}$ and $G\colon {}_A\mathfrak{M} \to {}_B\mathfrak{M}$ with $F \cdot G \simeq 1_{{}_A\mathfrak{M}}$ and $G \cdot F \simeq 1_{{}_B\mathfrak{M}}$. Knauer [3] (and independently, Banaschewski [1]) has shown that when A and B are Morita equivalent, there is an indecomposable projective generator Q in ${}_B\mathfrak{M}$ with $Q \simeq Bb$ where $b^2 = b \in B$ and $1, 1' \in B$ with $b1 = 1$ and $1'1 = e$, the identity of B. Moreover, $A \simeq F(Q)$, $A \simeq \mathrm{Hom}_B(Q, Q) \simeq bBb$ and $G(X) = {}_BQ_A \otimes X$.

In the following section, we show that when S is a monoid with 0 and ${}_SM, {}_SN \in {}_S\mathfrak{M}$ are retracts of each other, the endomorphism semigroups $\mathrm{End}_S(M)$ and $\mathrm{End}_S(N)$ are Morita equivalent. In §2, we use this result to show that when A and B are Morita equivalent monoids, then the McMorris quotient semigroups [7] $Q(A)$ and $Q(B)$ are Morita equivalent. This result was established in [5] for rings.

1. Retracts and Morita equivalence. Let S be a monoid and ${}_SM$ and ${}_SN$ be left S-systems. Set $A = \mathrm{End}_S(M)$ and $B = \mathrm{End}_S(N)$; then A and B operate on the right of M and N respectively and so we have bi-systems ${}_SM_A$, ${}_SN_B$, ${}_A\mathrm{Hom}_S(M, N)_B$ and ${}_B\mathrm{Hom}_S(N, M)_A$.

Now we assume that the S-system N is a retract of M by means of the mappings $M \underset{\alpha}{\overset{\beta}{\rightleftarrows}} N$ with $\alpha\beta = 1_N$. Define the mapping $\mu\colon \mathrm{Hom}_S(N, M) \otimes_A \mathrm{Hom}(M, N) \to B$ by $\mu(f \otimes g) = fg$. Then for $b \in B$, $\mu(b\alpha \otimes \beta) = b$ so μ is onto. Moreover if $\mu(f \otimes g) = \mu(x \otimes y)$, then $fg = xy$ so $f \otimes g = f \otimes g\alpha\beta = fg\alpha \otimes \beta = xy\alpha \otimes \beta = x \otimes y\alpha\beta = x \otimes y$. Thus μ is an isomorphism of (B, B)-systems.

Next define $\psi\colon \mathrm{Hom}_S(M, N) \to \mathrm{Hom}_A(\mathrm{Hom}_S(N, M), A)$ by $\psi(f)(g) = fg$ and $\psi'\colon \mathrm{Hom}_S(N, M) \to \mathrm{Hom}_A(\mathrm{Hom}_S(M, N), A)$ by $\psi'(g)(f) = fg$. Then ψ is an (A, B)-homomorphism and ψ' is a (B, A)-homomorphism. Using ψ we have a commutative diagram

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$$\begin{array}{ccc} M \otimes_A \mathrm{Hom}_S(M, N) & \overset{1\otimes\psi}{\rightarrow} & M \otimes_A \mathrm{Hom}_A(\mathrm{Hom}_S(N, M), A) \\ \tau\downarrow & & \downarrow\nu \\ N & \overset{i}{\rightarrow} & \mathrm{Hom}_A(\mathrm{Hom}_S(N, M), M) \end{array}$$

FIGURE 1

where $\tau(m \otimes f) = mf$, $i(n)(g) = ng$ and $\nu(m \otimes h)(g) = m \cdot (h(g))$ respectively. These are all (A, B)-homomorphisms.

THEOREM. *If ${}_SN$ is a retract of ${}_SM$, then*:

(1) *ψ and ψ' are isomorphisms.*

(2) *All homomorphisms in Figure* 1 *are isomorphisms.*

(3) $\mathrm{Hom}_S(M, N)$ *is a cyclic projective left A-system and a generator as a right B-system.*

(4) $\mathrm{End}_A(\mathrm{Hom}_S(M, N)) \simeq B$ *and* $\mathrm{End}_A(\mathrm{Hom}_S(N, M)) \simeq B$.

PROOF. (1) The inverse of ψ is $\psi^{-1}(t) = t(\alpha)\beta \in \mathrm{Hom}_S(M, N)$.

(2) Define k: $\mathrm{Hom}_A(\mathrm{Hom}_S(N, M), M) \to M \otimes_A \mathrm{Hom}_S(M, N)$ by $k(f) = f(\alpha) \otimes \beta$. Then all composition maps from one term to itself are the identity.

(3) ${}_A\mathrm{Hom}_S(M, N)$ is a retract of A by $f \in \mathrm{Hom}_S(M, N) \mapsto f\alpha \in \mathrm{Hom}_S(M, M) = A$ and $g \in A = \mathrm{Hom}_S(M, M) \mapsto g\beta \in \mathrm{Hom}_S(M, N)$. Likewise, $\mathrm{Hom}_S(N, N) = B$ is a retract of $\mathrm{Hom}_S(M, N)_B$ by $f \in \mathrm{Hom}_S(N, N) \mapsto \beta f \in \mathrm{Hom}_S(M, N)$ and $g \in \mathrm{Hom}_S(M, N) \mapsto \alpha g \in \mathrm{Hom}_S(N, N)$. Thus, $\mathrm{Hom}_S(M, N)$ is a cyclic projective left A-system and a generator as a right B-system.

(4) Let $\mathrm{Hom}_S(M, N) = At$ and since At is a retract of A, we can regard $at \neq bt$ for $a \neq b$. Now map $\mathrm{Hom}_A(At, A) \otimes_A At$ to $\mathrm{Hom}_A(At, At)$ by $f \otimes_A bt \mapsto [f, bt]$ where $(at)[f, bt] = (at)fbt$. This mapping is onto since each g: $At \to At$ is determined by its image $g(t) = a_g$, so that if h: $At \to A$ is given by $h(t) = a_g$, then $[h, t] = g$. To see that this mapping is injective, let $[h, bt] = [k, ct]$ so that $(th)bt = (tk)ct$, or $(th)b = (tk)c$ or $(t)hb = (t)kc$ and so $hb = kc$. Thus $h \otimes bt = hb \otimes t = kc \otimes t = k \otimes ct$ and so $\mathrm{Hom}_A(\mathrm{Hom}_S(M, N), A) \otimes_A \mathrm{Hom}_S(M, N)$ is isomorphic to $\mathrm{End}_A(\mathrm{Hom}_S(M, N))$. Define F: ${}_B\mathfrak{M} \to {}_A\mathfrak{M}$ by $F(X) = \mathrm{Hom}_S(M, N) \otimes_B X$ and G: ${}_A\mathfrak{M} \to {}_B\mathfrak{M}$ by $G(Y) = \mathrm{Hom}_S(N, M) \otimes_A Y$. Then $G \cdot F \simeq 1_{B\mathfrak{M}}$. Thus we have the following

THEOREM. *Let ${}_SM$ and ${}_SN$ be retracts of each other. Then* (1) $\mathrm{Hom}_S(M, N)$ *is a cyclic projective generator both as a left A-system and a right B-system, and* (2) *F and G are inverse equivalences between ${}_A\mathfrak{M}$ and ${}_B\mathfrak{M}$.*

2. Semigroups of quotients. A *special left quotient filter* Σ of a monoid B is a nonempty collection of left ideals of B saitsfying

Q1. $I, J \in \Sigma, f \in \mathrm{Hom}_B(I, B) \Rightarrow f^{-1}(J) \in \Sigma$;

Q2. if $J \in \Sigma$, $Ia^{-1} = \{s \in B | sa \in I\} \in \Sigma$ for all $a \in J$, then $I \in \Sigma$; and

Q3. $B \in \Sigma$.

For each left B-system M we define the torsion congruence τ_M by $m\tau_M n$ if there is $I \in \Sigma$ with $bm = bn$ for all $b \in I$. M is *strongly torsion free* if τ_M is the identity, and M is *torsion* if $\tau_M = \omega$. M is Σ-injective if whenever C is a subsystem of D and D/C is torsion, then each B-homomorphism f: $C \to M$ has an extension f: $D \to M$. Each strongly torsion free B-system has a unique (up to isomorphism over M) extension $E_\Sigma(M)$ which is the minimal Σ-injective extension of M. $E_\Sigma(M)$ is called the Σ-*injective hull* of M.

Given a special left quotient filter Σ, let $\mathfrak{B} = \bigcup_{I \in \Sigma} \mathrm{Hom}_B(I, B)$ and let θ be a congruence on β given by $f\theta g$ if there is some $J \in \Sigma$ on which f and g are both defined and agree. Then $Q_\Sigma(B) = \mathfrak{B}/\theta$ with the operation of functional composition fg: $g^{-1}(D_f) \cap D_g \to B$ forms a monoid. Moreover, $Q_\Sigma(B)$ is the Σ-injective hull of B if B is strongly torsion free. For other definitions and concepts we refer the reader to [4].

Let A and B be Morita equivalent monoids. We use the following notation to describe this situation.

(1) F: ${}_B\mathfrak{M} \to {}_A\mathfrak{M}$ and G: ${}_A\mathfrak{M} \to {}_B\mathfrak{M}$ are inverse category equivalences.

(2) ${}_AP_B$ is an (A, B)-bisystem which is a left A and right B progenerator (i.e., cyclic projective generator).

(3) ${}_BQ_A$ is a (B, A)-bisystem which is a left B and right A progenerator.

(4) $F\_ = P \otimes_{B}\_$ and $G\_ = Q \otimes_{A}\_$.

(5) $A \simeq \mathrm{End}_B(P) \simeq \mathrm{End}_B(Q)$ and $B \simeq \mathrm{End}_A(Q) \simeq \mathrm{End}_A(P)$.

(6) ${}_BQ \simeq \mathrm{Hom}_B(P, B)$ and ${}_AP \simeq \mathrm{Hom}_A(Q, A)$.

We have over-determined our equivalence. For A and B to be Morita equivalent it is necessary and sufficient that $A \simeq \mathrm{End}_B(P)$ for some progenerator P_B [5]. (Although [5] deals only with the additive case, since P_B is a cyclic projective generator, simple modifications of the existing proofs suffice to justify that $A \simeq \mathrm{End}_B(P)$ suffices for A and B to be Morita equivalent.) Since ${}_BQ$ is a progenerator and B is a monoid, B and Q are retracts of each other. Without loss of generality, since B is a monoid, Q is indecomposable [3].

Let Σ be a special left quotient filter on B and $\mathfrak{T}(B)$ be the class of all Σ-torsion B-systems. Σ uniquely determines and is determined by $\mathfrak{T}(B)$, where $\mathfrak{T}(B)$ is *special* in the sense that if $f \in \mathrm{Hom}(M, N)$ is 0-restricted (i.e., $f^{-1}(0) = \{0\}$) and $N \in \mathfrak{T}(B)$ then $M \in \mathfrak{T}(B)$ [4]. Let $\mathfrak{T}(A) = \{N \in {}_A\mathfrak{M} | G(N) \in \mathfrak{T}(B)\}$. Since G is an equivalence, $\mathfrak{T}(A)$ is closed under quotients, disjoint unions, and extensions and so is a torsion class for ${}_A\mathfrak{M}$. Moreover, $\mathfrak{T}(A)$ is a special torsion class, for if $f \in \mathrm{Hom}_A(M, N)$ is 0-restricted and $N \in \mathfrak{T}(A)$, then $G(N) \in \mathfrak{T}(B)$ and $G(f)$: $G(M) \to G(N)$. Suppose $G(K) \subset G(M)$ with $G(f)(G(K)) = 0$; then since G is an equivalence, $f(K) = 0$ so $K = 0$ since f is 0-restricted. Thus $G(K) = 0$ and $G(f)$ is 0-restricted. Since $\mathfrak{T}(B)$ is a special torsion class, $G(M) \in \mathfrak{T}(B)$ so $M \in \mathfrak{T}(A)$. Thus $\mathfrak{T}(A)$ is special and determines a unique special left quotient filter Γ on A by $\Gamma = \{J | G(A/J) \in \mathfrak{T}(B)\}$. Thus we have

THEOREM. $\Gamma = \{J | A/J \in \mathfrak{T}(A)\}$ *is a special left quotient filter on* A.

Now let $M \in {}_B\mathfrak{M}$; then since F and G are inverse equivalences, M is Σ-injective if and only if $F(M)$ is Γ-injective. This yields the following

PROPOSITION. $F(E_\Sigma(M)) = E_\Gamma(F(M))$.

Since ${}_BQ$ is a finitely generated projective generator, ${}_BQ$ and ${}_BB$ are retracts of each other. Knauer [3] (and independently Banaschewski [1]) has shown that $Q \simeq Bb$ where $b^2 = b \in B$. Now if B is strongly torsion free, the isomorphism between Q and Bb shows that Q is strongly torsion free. Then by the definition of Γ and the fact that F and G are category equivalences, A is strongly torsion free.

Now since ${}_BQ$ and ${}_BB$ are retracts of each other, $E_\Sigma(Q)$ and $E_\Sigma(B)$ are retracts of each other. Since $F(Q) \simeq_A A$ we have

$$\operatorname{End}_B(E_\Sigma(Q)) \simeq \operatorname{End}_A(F(E_\Sigma(Q))) \simeq \operatorname{End}_A(E_\Gamma(F(Q))) \simeq \operatorname{End}_A(E_\Gamma(A))$$

but $Q_\Sigma(B) \simeq \operatorname{End}_B(E_\Sigma(B))$ and $Q_\Gamma(A) \simeq \operatorname{End}_A(E_\Gamma(A)) \simeq \operatorname{End}_B(E_\Sigma(Q))$.

Let $M = E_\Sigma(Q)$ and $N = E_\Sigma(B)$; then from the last section we see that $Q_\Sigma(B)$ and $Q_\Gamma(A)$ are Morita equivalent and $\operatorname{Hom}_B(E_\Sigma(Q), E_\Sigma(B))$ is a cyclic projective generator both as a left $Q(A)$-system and right $Q(B)$-system.

Since $P \otimes_B E_\Sigma(B)$ is Γ-injective and $E_\Gamma(A)/A \in \mathfrak{T}(A)$,

$$\operatorname{Hom}_A(A, P \otimes_B E_\Sigma(B)) \simeq \frac{\operatorname{Hom}_A(E_\Gamma(A), P \otimes_B E_\Sigma(B))}{\operatorname{Hom}_A(E_\Gamma(A)/A, P \otimes_B E_\Sigma(B))},$$

but since $P \otimes_B E_\Sigma(B)$ is strongly torsion free,

$$\operatorname{Hom}_A(E_\Gamma(A)/A, P \otimes_B E_\Sigma(B)) = 0,$$

and $\operatorname{Hom}_A(A, P \otimes_B E_\Sigma(B))$ and $\operatorname{Hom}(E_\Gamma(A), P \otimes_B E_\Sigma(B))$ are isomorphic as right $Q_\Sigma(B)$-systems. Thus

$$\begin{aligned}\operatorname{Hom}_B(E_\Sigma(Q), E_\Sigma(B)) &\simeq \operatorname{Hom}_A(E_\Gamma(A), P \otimes_B E_\Sigma(B)) \simeq \operatorname{Hom}_A(A, P \otimes_B E_\Sigma(B)) \\ &\simeq P \otimes_B E_\Sigma(B) \simeq P \otimes_B Q_\Sigma(B).\end{aligned}$$

Thus we have proved the following

THEOREM. *Let Σ be a special left quotient filter on the monoid B and let A and B be Morita equivalent monoids. If Γ is the corresponding special left quotient filter on A, then $Q_\Gamma(A)$ and $Q_\Sigma(B)$ are Morita equivalent. Moreover, in the situation above, $P \otimes_B Q_\Sigma(B)$ is right $Q_\Sigma(B)$-cyclic projective generator and*

$$Q_\Gamma(A) \simeq \operatorname{End}_{Q_\Sigma(B)}(P \otimes_B Q_\Sigma(B)).$$

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