PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, October 1980

THE SCHNEIDER-LANG THEOREM FOR FUNCTIONS WITH ESSENTIAL SINGULARITIES

JACK DIAMOND

ABSTRACT. A new proof of Schwarz's lemma for functions with a finite number of essential singularities is given. The proof is valid for p-adic as well as complex functions and is used to extend Bertrand's version of the Schneider-Lang theorem for p-adic functions with one, common, finite singularity to functions with finitely many singularities.

1. Introduction. The Schneider-Lang theorem gives a maximum for the number of points at which a family of meromorphic functions satisfying certain conditions can simultaneously take on values in a number field. D. Bertrand has extended it to include the case of a family of complex functions having a finite set of essential singularities and to the case of a family of p-adic functions having a common essential singularity. These generalizations are based on an extension of Schwarz's lemma to functions with singularities. Schwarz's lemma is a refinement of the maximum modulus principle in the case where a function is known to have many zeroes.

We have another, more algebraic, approach to Schwarz's lemma that only uses analytic concepts which apply to both complex and p-adic functions. Consequently, we will give a unified proof of the Schneider-Lang theorem which is more elementary than Bertrand's in the complex case and extends his result in the p-adic case.

$\mathbf{C}$ is the field of complex numbers and $\mathbf{C}_p$ is the completion of the algebraic closure of the p-adic field $\mathbf{Q}_p$. $\mathbf{C}'$ will denote both $\mathbf{C} \cup \{\infty\}$ and $\mathbf{C}_p \cup \{\infty\}$. $|\mathbf{C}'|$ is the set of values of $|x|$ for $x \in \mathbf{C}'$, $x \neq \infty$. If $\mathcal{Q} = \{a_1, \ldots, a_k\}$ is a finite subset of $\mathbf{C}'$, then as a concept of analyticity on $\mathbf{C}' - \mathcal{Q}$ we use the property that $f(x) = \Sigma_{1<i<k} f_i(x)$ where $f_i(x)$ is a power series in x if $a_i = \infty$ and a power series in $(x - a_i)^{-1}$ if $a_i \neq \infty$. In the p-adic case this is not as general as Krasner's concept of analyticity. The restriction is made in order to use the maximum modulus principle on disc-shaped domains, a principle, which, as is shown in [4], does not apply to (Krasner) analytic functions.

$C(a, r)$ is the circle with center a and radius r if $a \neq \infty$ and the circle with center 0 and radius $1/r$ if $a = \infty$. If $a = \infty$ the usual exterior of $C(0, 1/r)$ will be called the interior. Given a family of circles $C(a_i, r_i)$ in which each r_i is a function of r, we let $\mathfrak{D}(r) = \bigcup_i C(a_i, r_i)$ and $m_f(r) = \sup_{x \in \mathfrak{D}(r)} |f(x)|$.

Received by the editors November 26, 1979; presented to the Society, August 24, 1979.
AMS (MOS) subject classifications (1970). Primary 10F35, 12B40.
Key words and phrases. Schwarz's lemma, Schneider-Lang theorem.

2. Schwarz's lemma. This version of Schwarz's lemma is similar to Bertrand's complex version in [1] and extends his p-adic version in [2] to functions with finitely many singularities.

THEOREM (SCHWARZ'S LEMMA). *Suppose that f is analytic on $\mathbf{C}' - \mathfrak{A}$, $\mathfrak{A} = \{a_1, \ldots, a_k\}$, $f \not\equiv 0$, f has zeroes at $z_1, \ldots, z_h$ (not necessarily distinct), $\mathfrak{Z}$ is the set of distinct elements among $z_1, \ldots, z_h$, $\mathfrak{W} = \{w_1, \ldots, w_k\}$ is a set of positive rational numbers whose sum is W, $R_i = R^{1/w_i}$ and r is chosen small enough that the circles $C(a_i, r_i)$ do not have any points of $\mathfrak{Z}$ or of $\mathfrak{A}$, excepting the given center, on or within themselves. Then, if R and $r \in |\mathbf{C}'|$ and $0 < R \leqslant r$, there is a constant c depending only on $\mathfrak{A}$, $\mathfrak{W}$, $\mathfrak{Z}$ and r so that*

$$m_f(r) < c^h (R/r)^{h/W} m_f(R).$$

PROOF. First, suppose that $\mathfrak{W}$ is a set of positive integers. Then, $f^W(x) = g(x)\varphi(x)$ where

$$\varphi(x) = \frac{\prod^*_{1 \leqslant j \leqslant h}(x - z_j)^W}{\prod^*_{1 \leqslant i \leqslant k}(x - a_i)^{hw_i}}.$$

The * in the expression for φ indicates that if a_i or z_j is ∞ the corresponding factor should be omitted.

The point of this factorization for $f^W(x)$ is that g is also analytic on $\mathbf{C}' - \mathfrak{A}$ and $|\varphi(x)|$ can be easily estimated.

We will find upper and lower bounds for $|\varphi(x)|$ and then apply the maximum modulus principle to g to obtain Schwarz's lemma. The maximum modulus principle in this situation is that $m_g(r) \leqslant m_g(R)$. In the p-adic case this is a consequence of the maximum modulus principle on discs.

In order to estimate $|\varphi(x)|$, $x \in \mathfrak{D}(R)$, it is necessary to consider $a_i = \infty$ separately from the other a_i. r will be fixed throughout and $R \leqslant r$.

Case 1. $a_I \neq \infty$, $x \in C(a_I, R_I)$,

$$|\varphi(x)| = \frac{R^{-h}\prod^*_{1 \leqslant j \leqslant h}|x - z_j|^W}{\prod^*_{1 \leqslant i \leqslant k, i \neq I}|x - a_i|^{hw_i}}.$$

Clearly, $|x - z_j|$ and $|x - a_j|$ are bounded above by a number depending only on $\mathfrak{A}$, $\mathfrak{Z}$, $\mathfrak{W}$ and r.

On the other hand, $|x - z_j|$ and $|x - a_i|$ are bounded below by $m - r_I$, where m is the minimum distance between two finite points in $\mathfrak{A} \cup \mathfrak{Z}$.

The next step depends on whether or not $\infty \in \mathfrak{A}$. The calculations are similar, though the possibility of $z_j = \infty$ needs to be taken into account if ∞ is not a singular point. Suppose $a_J = \infty$ and M is the maximum distance between two finite points in $\mathfrak{A} \cup \mathfrak{Z}$. Then

$$\frac{R^{-h}(m - r_I)^{hW}}{(M + r_I)^{h(W - w_I - w_J)}} < |\varphi(x)| < \frac{R^{-h}(M + r_I)^{hW}}{(m - r_I)^{h(W - w_I - w_J)}}$$

and $0 < R^{-h}A_I^h < |\varphi(x)| < R^{-h}B_I^h$ with

$$A_I = \frac{(m - r_I)^W}{(M + R_I)^{W - w_I - w_J}}, \qquad B_I = \frac{(M + r_I)^W}{(m - r_I)^{W - w_I - w_J}}.$$

Case 2. $a_I = \infty$, $x \in C(\infty, R_I)$,

$$|\varphi(x)| = \frac{\prod_{1 \leq j \leq h}^{*} |x - z_j|^W}{\prod_{1 \leq i \leq k}^{*} |x - a_i|^{hw_i}}.$$

Let $M = \max\{|a_i|, |z_j| \mid a_i \neq \infty\}$. Then

$$1/r_I - M < |x - z_j| < 1/R_I + M, \qquad 1/r_I - M < |x - a_i| < 1/R_I + M$$

and

$$\frac{(1/R_I - M)^{hW}}{(1/R_I + M)^{h(W - w_I)}} < |\varphi(x)| < \frac{(1/R_I + M)^{hW}}{(1/R_I - M)^{h(W - w_I)}}.$$

Hence, $0 < R^{-h} A_I^h < |\varphi(x)| < R^{-h} B_I^h$ with

$$A_I = \frac{(1 - Mr_I)^W}{(1 + Mr_I)^{W - w_I}}, \qquad B_I = \frac{(1 + Mr_I)^W}{(1 - Mr_I)^{W - w_I}}.$$

If $A = \min_I\{A_I\}$ and $B = \max_I\{B_I\}$, then when $x \in \mathfrak{D}(R)$

$$0 < R^{-h} A^h < |\varphi(x)| < R^{-h} B^h.$$

Since $f^W(x) = g(x)\varphi(x)$, it follows that

$$R^{-h} A^h |g(x)| \leq |f^W(x)| \leq R^{-h} B^h |g(x)|$$

and

$$0 < R^{-h} A^h m_g(R) < m_{f^W}(R) < R^{-h} B^h m_g(R).$$

Hence, since $R \leq r$,

$$\begin{aligned} m_{f^W}(r) &< r^{-h} B^h m_g(r) \leq r^{-h} B^h m_g(R) \\ &= (R/r)^h (B/A)^h A^h R^{-h} m_g(R) < (R/r)^h (B/A)^h m_{f^W}(R). \end{aligned}$$

Choosing $c = (B/A)^{1/W}$ and taking the Wth root of each side of the above inequality yields

$$m_f(r) < (R/r)^{h/W} c^h m_f(R).$$

If the w_i are rational numbers with a common denominator of d, then the $R_i = R^{1/w_i}$ have the form $(R^d)^{1/u_i}$ with u_i an integer. The theorem now follows from the case for integral u_i already proved.

3. The Schneider-Lang theorem. Before stating the Schneider-Lang theorem we will recall the definition for finite order of growth of a function at an isolated singular point. If $a \neq \infty$, then f has order of growth $\leq w$ at a if there is a neighborhood around a and a constant A so that $|f(x)| < \exp(AR^{-w})$ when $|x - a| = R$. If $a = \infty$, then f has order of growth $\leq w$ at a if there is a neighborhood around a and a constant A so that $|f(x)| < \exp(AR^w)$ when $|x| = R$.

If $\mathcal{A} = \{a_1, \ldots, a_k\}$ is a finite subset of $\mathbf{C}'$ and $\mathcal{W} = \{w_1, \ldots, w_k\}$ is a set of positive real numbers, let $\mathfrak{M}(\mathcal{A}, \mathcal{W})$ be the field of fractions of the ring of functions which are analytic on $\mathbf{C}' - \mathcal{A}$ and have order $\leq w_i$ at a_i, $i = 1, \ldots, k$.

The Schneider-Lang theorem is given in [3] for functions meromorphic on **C**.

Bertrand generalized the theorem to complex functions with a finite number of essential singularities in [1] and to p-adic functions with one common finite singularity in [2]. It will be shown below that Bertrand's result for complex functions also holds for p-adic functions.

THEOREM. *Suppose that $\mathcal{Q}$ is a set of k points of $\mathbf{C}'$, $f_1, \ldots, f_N$ are mappings from $\mathbf{C}' - \mathcal{Q}$ into $\mathbf{C}'$, $\mathcal{W}$ is a set of k positive rational numbers whose sum is W, $\mathbf{K}$ is a number field embedded in $\mathbf{C}'$ and the following conditions are satisfied.*

(i) f_1 and f_2 are algebraically independently over $\mathbf{K}$.

(ii) d/dx maps $\mathbf{K}[f_1, \ldots, f_N]$ into itself.

(iii) There is a set of distinct points $z_1, \ldots, z_m$ such that $f_i(z_j) \in \mathbf{K}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, m$.

(iv) $f_i \in \mathfrak{M}(\mathcal{Q}, \mathcal{W})$ for $i = 1, \ldots, N$.

Then $m \leqslant 2W[\mathbf{K} : \mathbf{Q}]$.

The proof is a standard argument which appears in several of the references. We will give an outline to show how Schwarz's lemma is applied.

Given a positive integer S, a function F_S, not identically zero, is constructed as a polynomial in f_1 and f_2. F_S has the property that it has a zero of order at least S at each z_j, $j = 1, \ldots, m$. Suppose that for each j $D^n F_S(z_j) = 0$ if $n < \sigma$ and for some J $D^\sigma F_S(z_J) \neq 0$. $\sigma \geqslant S$. Let $\gamma_\sigma = D^\sigma F_S(z_J)/\sigma!$.

On the basis of hypotheses (i), (ii) and (iii) a lower bound for $\log|\gamma_\sigma|$ can be obtained,

$$\log|\gamma_\sigma| \geqslant -([\mathbf{K} : \mathbf{Q}] + o(1))\sigma \log \sigma$$

where $o(1)$ approaches 0 as σ approaches ∞.

In order to obtain an upper bound for $\log|\gamma_\sigma|$, Cauchy's theorem is used to change the problem to that of bounding $|F_S(x)|$ on a small circle around z_J. Now, Schwarz's lemma with r fixed, $R = \sigma^{-1/2}$, $R_i = R^{1/w_i}$ and $h = m\sigma$ is used together with hypothesis (iv) to show that

$$\log|\gamma_\sigma| \leqslant (-m/2W)\sigma \log \sigma + o(\sigma \log \sigma).$$

When the upper and lower bounds for $\log|\gamma_\sigma|$ are compared we find $m \leqslant 2W[\mathbf{K} : \mathbf{Q}]$.

REFERENCES

1. D. Bertrand, *Un théorème de Schneider-Lang sur certains domaines non simplement connexes*, Séminaire Delange-Pisot-Poitou (16e année: 1974/75), Théorie des Nombres, Fasc. 2, Exp. No. G-18, Secrétariat Mathématique, Paris, 1975.

2. ______, *Séries d'Eisenstein et transcendance*, Bull. Soc. Math. France, **104** (1976), 309–321.

3. S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, Reading, Mass., 1966.

4. P. Robba and A. Escassut, *Prolongement analytique et algèbres de Banach ultrametrique*, Astérisque **10**, Soc. Math. France, 1973.

5. M. Waldschmidt, *Nombres transcendants*, Lecture Notes in Math., vol. 402, Springer-Verlag, Berlin, 1974.

DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE, FLUSHING, NEW YORK 11367