STRONG LIFTINGS WHICH ARE NOT BOREL LIFTINGS

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ABSTRACT. The purpose of this note is to prove that any strong lifting of the circle commuting with translations cannot be a Borel lifting.

1. Introduction. In [4], A. and C. Ionescu-Tulcea proved that, if \( G \) is a locally compact topological group with left Haar measure \( \gamma \), then \( L^\infty(G, \gamma) \) admits a strong lifting \( \rho \) which commutes with left translations. In [7], von Neumann and Stone proved (using the continuum hypothesis) that, if \( X \) is locally compact metric and \( \mu \) is a positive Radon measure on \( X \), then \( L^\infty(X, \mu) \) admits a Borel lifting \( \rho \) (i.e., \( \rho(f) \) is a Borel function whenever \( f \in L^\infty(X, \mu) \)).

It is natural to ask whether a strong lifting \( \rho \) on \( G \) commuting with translations is a Borel lifting. We will prove that if \( G = K = \text{circle} \), then \( \rho \) cannot be a Borel lifting. The proof is a simple combination of techniques of topological dynamics due to Furstenberg [3] and Veech [9].

2.

2.1. Definitions. Let \( X \) be a locally compact Hausdorff space, and let \( \mu \) be a positive Radon measure on \( X \). Let \( M^\infty(X, \mu) = \{ f: X \to \mathbb{C} | f \text{ is bounded and } \mu\text{-measurable} \} \). A map \( \rho: M^\infty(X, \mu) \to M^\infty(X, \mu) \) is a linear lifting of \( M^\infty(X, \mu) \) if

(i) \( \rho(f) = f \mu\text{-a.e.} \),
(ii) \( f = g \) locally \( \mu\text{-a.e.} \Rightarrow \rho(f) = \rho(g) \) everywhere,
(iii) \( \rho \) is linear,
(iv) \( \rho(1) = 1 \),
(v) \( f > 0 \Rightarrow \rho(f) > 0 \) (\( f, g \in M^\infty(X, \mu) \)).

If, in addition, (vi) \( \rho(f \cdot g) = \rho(f) \cdot \rho(g) \) for all \( f \) and \( g \), then \( \rho \) is a lifting. If (vii) \( \rho(f) = f \) for all bounded continuous \( f: X \to \mathbb{C} \), and if (i)-(vi) are satisfied, then \( \rho \) is a strong lifting of \( M^\infty(X, \mu) \).

2.2. Theorem [4]. Let \( G \) be a locally compact topological group with left Haar measure \( \gamma \). There is a strong lifting \( \rho \) of \( M^\infty(G, \gamma) \) which commutes with left translations. That is, if \( f \in M^\infty(X, \mu) \), \( g \in G \), and \( (f \cdot g)(x) = \text{def} f(g \cdot x) \) (\( x \in G \)), then \( \rho(f \cdot g) = \rho(f) \cdot g \) (\( g \in G \)).

2.3. Theorem [7, Theorem 17]. Let \( X \) be a locally compact metric space with positive Radon measure \( \mu \). Then there is a Borel lifting \( \rho \) of \( M^\infty(X, \mu) \). That is, if \( f \in M^\infty(X, \mu) \), then \( \rho(f) \) is Borel measurable.

As is remarked in [5], it is not known whether the above theorem may be proved without using the continuum hypothesis.

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2.4. Definitions and remarks. Let $K$ be the circle group, viewed as the unit circle in $\mathbb{C}$. Let $\gamma$ be normalized Haar measure on $K$, and let $\rho$ be a strong lifting of $M^\infty(K, \gamma)$ commuting with translations. Let $\alpha_0$ be some irrational number and let $\alpha = e^{2\pi i \alpha_0}$. Then $\{\alpha^n : n \in \mathbb{Z}\}$ is dense in $K$. Using a technique of Furstenberg [3, §3], we may construct a function $r : K \to K$ such that (i) $r$ is $\gamma$-measurable, but is not equal $\gamma$-a.e. to a continuous function on $K$, (ii) $r(\omega \cdot \alpha)/r(\omega)$ is equal $\gamma$-a.e. to a continuous function $q$ on $K$. See also [6]. Note that, if $\bar{r} = \rho(r)$, then $|\bar{r}(\omega)| = 1$ and $\bar{r}(\omega \cdot \alpha)/\bar{r}(\omega) = q(\omega)$ for all $\omega \in \Omega$. Replacing $r$ by $\bar{r}$, we assume from now on that $\rho(r) = r$, and that $r(\omega \cdot \alpha)/r(\omega) = q(\omega)$ for all $\omega \in \Omega$.

2.5. Definitions. Let $X$ be a compact Hausdorff space, and let $T$ be a topological group. A flow on $X$ is defined by a triple $(X, T, \eta)$, where $\eta : X \times T \to X : (x, t) \mapsto x \cdot t$ is a continuous map such that (i) $x \cdot e = x$ ($x \in X$, $e =$ identity in $T$), (ii) $(x \cdot t_1) \cdot t_2 = x \cdot (t_1 t_2)$ ($x \in X$, $t_1, t_2 \in T$). We suppress $\eta$, and denote a flow by $(X, T)$. A subset $A$ of $X$ is invariant if $A \supseteq A \cdot T = \{x \cdot t \mid x \in A, t \in T\}$. A flow $(X, T)$ is minimal if each orbit $\{x \cdot t \mid t \in T\} \subseteq X$ is dense in $X (x \in X)$.

2.6. Definition. Let $K, r, q, \alpha$ be as in 2.4. Define a homeomorphism $h$ of $K \times K = K^2$ as follows: $h(w, r) = (w \cdot r, q(w \cdot r))$. Then $h$ defines a flow $(K^2, Z)$ in the obvious way: $(w \cdot r) \cdot n = h^n(w, r)$, where $h^n$ is the $n$-fold composition ($n \in \mathbb{Z}$). (We give $Z$ the discrete topology.)

The following proposition is a corollary of [1, Lemma 1.9 and Theorem 1.17].

2.7. Proposition. The flow $(K^2, Z)$ defined in Definition 2.6 is minimal.

2.8. Definition [8, Definition 0.2]. Let $(K, Z)$ be the flow defined by $\omega \cdot n = \omega \cdot \alpha^n$ ($\omega \in K, n \in \mathbb{Z}$). Let $(X, Z)$ be another minimal flow, suppose $\pi : X \to K$ is continuous and suppose $\pi(x \cdot n) = \pi(x) \cdot n$ ($x \in X, n \in \mathbb{Z}$). Then $\pi$ is onto. We say $(X, Z)$ is an extension of $(K, Z)$. Say that $(X, Z)$ is an almost automorphic (a.a.) extension of $(K, Z)$ if $\pi^{-1}(\omega)$ is a singleton for some $\omega \in K$.

The following theorem is a special case of [9, Proposition 2.3.9].

2.9. Theorem. Let $(X, Z)$ be an extension of $(K, Z)$ with $X$ compact metric. Suppose there is an invariant Borel subset $Q$ of $X$ such that $Q \cap \pi^{-1}(\omega)$ is a singleton for each $\omega \in K$. Then $(X, Z)$ is an a.a. extension of $(K, Z)$.

2.10. Theorem. The strong lifting $\rho$ of $M^\infty(K, \gamma)$ is not a Borel lifting.

Proof. Suppose for contradiction that $\rho(f)$ is a Borel for all $f \in M^\infty(K, \gamma)$. Then $r = \rho(r)$ is Borel measurable. Hence the set $Q = \{(\omega, r(\omega)) : \omega \in K\}$ is a Borel subset of $K^2$. It is also invariant under the flow $(K^2, Z)$. By Theorem 2.9, $(K^2, Z)$ is an a.a. extension of $(K, Z)$. This is absurd. So $\rho(r)$ cannot be Borel.

2.11. Remarks. (1) The proof of Theorem 2.10 requires only that $\rho$ commute with multiplication by all the $\alpha^n$ ($n \in \mathbb{Z}$).

(2) It is plausible to conjecture that, if $G$ and $\rho$ are as in Theorem 2.2, then $\rho$ cannot be a Borel lifting. This statement is true if $G$ is metrizable and contains $Z$ as an embedded dense subgroup. For, in this case, a function $r$ with properties (i) and (ii) of 2.4 may be constructed.
References


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