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STRONG LIFTINGS WHICH ARE NOT BOREL LIFTINGS

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ABSTRACT. The purpose of this note is to prove that any strong lifting of the circle commuting with translations cannot be a Borel lifting.

1. Introduction. In [4], A. and C. Ionescu-Tulcea proved that, if G is a locally compact topological group with left Haar measure γ, then $L^\infty(G, \gamma)$ admits a strong lifting ρ which commutes with left translations. In [7], von Neumann and Stone proved (using the continuum hypothesis) that, if X is locally compact metric and μ is a positive Radon measure on X, then $L^\infty(X, \mu)$ admits a Borel lifting ρ (i.e., $\rho(f)$ is a Borel function whenever $f \in L^\infty(X, \mu)$).

It is natural to ask whether a strong lifting ρ on G commuting with translations is a Borel lifting. We will prove that if $G = K =$ circle, then ρ cannot be a Borel lifting. The proof is a simple combination of techniques of topological dynamics due to Furstenberg [3] and Veech [9].

2.

2.1. *Definitions*. Let X be a locally compact Hausdorff space, and let μ be a positive Radon measure on X. Let $M^\infty(X, \mu) = \{f\colon X \to \mathbf{C} \mid f$ is bounded and μ-measurable$\}$. A map $\rho\colon M^\infty(X, \mu) \to M^\infty(X, \mu)$ is a *linear lifting* of $M^\infty(X, \mu)$ if (i) $\rho(f) = f$ μ-a.e., (ii) $f = g$ locally μ-a.e.$\Rightarrow \rho(f) = \rho(g)$ *everywhere*, (iii) ρ is linear, (iv) $\rho(1) = 1$, (v) $f \geqslant 0 \Rightarrow \rho(f) \geqslant 0$ $(f, g \in M^\infty(X, \mu))$. If, in addition, (vi) $\rho(f \cdot g) = \rho(f) \cdot \rho(g)$ for all f and g, then ρ is a *lifting*. If (vii) $\rho(f) = f$ for all bounded continuous $f\colon X \to \mathbf{C}$, and if (i)–(vi) are satisfied, then ρ is a *strong lifting* of $M^\infty(X, \mu)$.

2.2. THEOREM [4]. *Let G be a locally compact topological group with left Haar measure γ. There is a strong lifting ρ of $M^\infty(G, \gamma)$ which commutes with left translations. That is, if $f \in M^\infty(X, \mu)$, $g \in G$, and $(f \cdot g)(x) =_{\text{def}} f(g \cdot x)$ $(x \in G)$, then $\rho(f \cdot g) = \rho(f) \cdot g$ $(g \in G)$.*

2.3. THEOREM [7, THEOREM 17]. *Let X be a locally compact metric space with positive Radon measure μ. Then there is a Borel lifting ρ of $M^\infty(X, \mu)$. That is, if $f \in M^\infty(X, \mu)$, then $\rho(f)$ is Borel measurable.*

As is remarked in [5], it is not known whether the above theorem may be proved without using the continuum hypothesis.

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2.4. *Definitions and remarks.* Let K be the circle group, viewed as the unit circle in $\mathbf{C}$. Let γ be normalized Haar measure on K, and let ρ be a strong lifting of $M^{\infty}(K, \gamma)$ commuting with translations. Let α_0 be some irrational number and let $\alpha = e^{2\pi i \alpha_0}$. Then $\{\alpha^n: n \in \mathbf{Z}\}$ is dense in K. Using a technique of Furstenberg [**3**, §3], we may construct a function $r: K \to K$ such that (i) r is γ-measurable, but is not equal γ-a.e. to a continuous function on K, (ii) $r(\omega \cdot \alpha)/r(\omega)$ is equal γ-a.e. to a continuous function q on K. See also [**6**]. Note that, if $\tilde{r} = \rho(r)$, then $|\tilde{r}(\omega)| = 1$ and $\tilde{r}(\omega \cdot \alpha)/\tilde{r}(\omega) = q(\omega)$ for *all* $\omega \in \Omega$. Replacing r by $\tilde{r}$, we assume from now on that $\rho(r) = r$, and that $r(\omega \cdot \alpha)/r(\omega) = q(\omega)$ for all $\omega \in \Omega$.

2.5. Definitions. Let X be a compact Hausdorff space, and let T be a topological group. A *flow on* X is defined by a triple (X, T, η), where η: $X \times T \to X$: $(x, t) \to x \cdot t$ is a continuous map such that (i) $x \cdot e = x$ ($x \in X$, e = identity in T), (ii) $(x \cdot t_1) \cdot t_2 = x \cdot (t_1 t_2)$ ($x \in X$, $t_1, t_2 \in T$). We suppress η, and denote a flow by (X, T). A subset A of X is *invariant* if $A \supset A \cdot T = \{x \cdot t \mid x \in A, t \in T\}$. A flow (X, T) is *minimal* if each orbit $\{x \cdot t \mid t \in T\} \subset X$ is dense in X ($x \in X$).

2.6. Definition. Let K, r, q, α be as in 2.4. Define a homeomorphism h of $K \times K \equiv K^2$ as follows: $h(\omega, \zeta) = (\omega \cdot \alpha, q(\omega)\zeta)$. Then h defines a flow $(K^2, \mathbf{Z})$ in the obvious way: $(\omega \cdot \zeta) \cdot n = h^n(\omega, \zeta)$, where h^n is the n-fold composition ($n \in \mathbf{Z}$). (We give $\mathbf{Z}$ the discrete topology.)

The following proposition is a corollary of [**1**, Lemma 1.9 and Theorem 1.17].

2.7. Proposition. *The flow* $(K^2, \mathbf{Z})$ *defined in Definition* 2.6 *is minimal.*

2.8. Definition [**8**, Definition 0.2]. Let $(K, \mathbf{Z})$ be the flow defined by $\omega \cdot n = \omega \cdot \alpha^n$ ($\omega \in K$, $n \in \mathbf{Z}$). Let $(X, \mathbf{Z})$ be another minimal flow, suppose π: $X \to K$ is continuous and suppose $\pi(x \cdot n) = \pi(x) \cdot n$ ($x \in X$, $n \in \mathbf{Z}$). Then π is onto. We say $(X, \mathbf{Z})$ is an *extension* of $(K, \mathbf{Z})$. Say that $(X, \mathbf{Z})$ is an *almost automorphic* (*a.a.*) *extension* of $(K, \mathbf{Z})$ if $\pi^{-1}(\omega)$ is a singleton for some $\omega \in K$.

The following theorem is a special case of [**9**, Proposition 2.3.9].

2.9. Theorem. *Let* $(X, \mathbf{Z})$ *be an extension of* $(K, \mathbf{Z})$ *with* X *compact metric. Suppose there is an invariant Borel subset* Q *of* X *such that* $Q \cap \pi^{-1}(\omega)$ *is a singleton for each* $\omega \in K$. *Then* $(X, \mathbf{Z})$ *is an a.a. extension of* $(K, \mathbf{Z})$.

2.10. Theorem. *The strong lifting* ρ *of* $M^{\infty}(K, \gamma)$ *is not a Borel lifting.*

Proof. Suppose for contradiction that $\rho(f)$ is a Borel for all $f \in M^{\infty}(K, \gamma)$. Then $r = \rho(r)$ is Borel measurable. Hence the set $Q = \{(\omega, r(\omega)): \omega \in K\}$ is a Borel subset of K^2. It is also invariant under the flow $(K^2, \mathbf{Z})$. By Theorem 2.9, $(K^2, \mathbf{Z})$ is an a.a. extension of $(K, \mathbf{Z})$. This is absurd. So $\rho(r)$ cannot be Borel.

2.11. Remarks. (1) The proof of Theorem 2.10 requires only that ρ commute with multiplication by all the α^n ($n \in \mathbf{Z}$).

(2) It is plausible to conjecture that, if G and ρ are as in Theorem 2.2, then ρ cannot be a Borel lifting. This statement is true if G is metrizable and contains $\mathbf{Z}$ as an embedded dense subgroup. For, in this case, a function r with properties (i) and (ii) of 2.4 may be constructed.

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