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BOHR COMPACTIFICATION AND CONTINUOUS MEASURES

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ABSTRACT. Let G be an LCA group with dual Γ. As a consequence of our main result, it is shown that every continuous regular measure μ concentrated on a Kronecker set and with norm > 1 has the property that $\{|\hat{\mu}| > 1\}$ is dense in the Bohr compactification of Γ.

Y. Katznelson [3] constructs a continuous measure μ on the circle group such that $\{n \in \mathbf{Z}: |\hat{\mu}(n)| > 1\}$ is dense in the Bohr compactification of $\mathbf{Z}$. In the present note we point out that every continuous measure (with norm > 1) concentrated on a Kronecker set has this property.

Let G be a *nondiscrete* LCA group with dual Γ, and let $M(G)$ be the convolution measure algebra of G (cf. [2]). For $\mu \in M(G)$, we denote by $\hat{\mu}$ the Fourier-Stieltjes transform of μ. It is easy to show that every measure $\mu \in M(G)$ concentrated on a Kronecker set (or a K_p-set) has the following property, which we call (HK): Given $\gamma \in \Gamma$, a Borel set E in G, and $\varepsilon > 0$, there exists $\chi \in \Gamma$ such that

$$\int_E |\gamma - \chi| \, d|\mu| + \int_{G \setminus E} |1 - \chi| \, d|\mu| < \varepsilon.$$

THEOREM. *Let $\mu \in M(G)$ be a continuous measure having property (HK), and let $\mu_1, \ldots, \mu_n \in M(G)$ be absolutely continuous with respect to μ. Then, for each nonempty (relatively) open subset U of $\{(\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)): \gamma \in \Gamma\} \subset \mathbf{C}^n$, the set*

$$\{\gamma \in \Gamma: (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$$

is dense in the Bohr compactification of Γ.

To prove this, we need two lemmas. Let $\mathbf{T} = \{z \in \mathbf{C}: |z| = 1\}$ be the circle group.

LEMMA 1. *Given $\varepsilon > 0$, there exists a natural number r having the following property. If $z_1, z_2, \ldots, z_N$ are finitely many elements of $\mathbf{T}$, there exist $k(1), k(2), \ldots, k(r) \in \{1, 2, \ldots, N\}$ and $p(1), p(2), \ldots, p(r) \in \mathbf{Z}$ such that*

$$|z_1 \cdots z_N - z_{k(1)}^{p(1)} \cdots z_{k(r)}^{p(r)}| < \varepsilon.$$

PROOF. Let S be the subgroup of $\mathbf{T}$ that is generated by all elements of $\mathbf{T}$ having order less than $2\pi/\varepsilon$. Then S is a finite group. We define r to be the order of S.

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Now let $z_1, \ldots, z_N \in \mathbf{T}$ be given. If $\operatorname{ord}(z_j) \geqslant 2\pi/\varepsilon$ for some index j, then $\{z_j^p: p \in \mathbf{Z}\}$ is ε-dense in $\mathbf{T}$. Therefore it sufficies to set $k(1) = k(2) = \cdots = k(r) = j$, $p(1) = p$ for some $p \in \mathbf{Z}$, and $p(2) = \cdots = p(r) = 0$. So assume that $\operatorname{ord}(z_k) < 2\pi/\varepsilon$ for all indices k. Then $\{z_k: 1 \leqslant k \leqslant N\}$ is contained in S and therefore consists of at most r different elements. Evidently this completes the proof.

LEMMA 2. *Suppose $\rho \in M(G)$ is a nonnegative continuous measure having property (HK), F is a finite set in G, $\gamma_0, \gamma_1 \in \Gamma$ and $\varepsilon > 0$. Then there exists $\gamma_2 \in \Gamma$ such that*

$$|\gamma_0 - \gamma_2| < \varepsilon \text{ on } F \quad \text{and} \quad \int |\gamma_1 - \gamma_2| \, d\rho < \varepsilon.$$

PROOF. Replacing γ_1 by $\bar{\gamma}_0\gamma_1$, we may assume that $\gamma_0 = 1$. Let us enumerate the elements of F as $x_1, \ldots, x_n$, and define $F_0 = \varnothing$ and $F_j = \{x_1, \ldots, x_j\}$ for $j = 1, 2, \ldots, n$. Put $\nu = \delta_1 + \cdots + \delta_n + \rho$, where δ_j is the unit point measure at x_j. For $\mu \in M(G)$, we denote by $\Gamma(\mu)$ the closure of Γ in $L^1(\mu)$. Notice that $\Gamma(\mu)$ forms a multiplicative group.

By induction on $j = 0, 1, \ldots, n$, we shall prove that given $f \in \Gamma(\rho)$, there exists $g \in \Gamma(\nu)$ such that $g = f$ a.e. $d\rho$ and $g = 1$ on F_j. Since F is a finite set and $F_0 = \varnothing$, this is obvious for $j = 0$. So assume that $1 \leqslant j \leqslant n$ and that the result is true for $j - 1$. Choose and fix any $f \in \Gamma(\rho)$ and any $\varepsilon > 0$.

Let $r = r(\varepsilon)$ be the natural number given by Lemma 1. Since ρ is a continuous measure, we can partition Γ into disjoint Borel sets $E_1, E_2, \ldots, E_N$ such that $\rho(E_k) < \varepsilon/(2r)$ for all $k = 1, 2, \ldots, N$. Without loss of generality, assume that $N > r$. Since ρ has property (HK) and $f \in \Gamma(\rho)$, it is obvious that there exist $f_1, f_2, \ldots, f_N$ in $\Gamma(\rho)$ such that $f_k = f$ a.e. $d\rho$ on E_k, and $f_k = 1$ a.e. $d\rho$ on $G \setminus E_k$. It follows from the inductive hypothesis that there exist $g_1, g_2, \ldots, g_N$ in $\Gamma(\nu)$ such that $g_k = f$ a.e. $d\rho$ on E_k, $g_k = 1$ a.e. $d\rho$ on $G \setminus E_k$, and $g_k = 1$ on F_{j-1}. We now apply Lemma 1 with $z_k = g_k(x_j)$ to find $k(1), \ldots, k(r) \in \{1, 2, \ldots, N\}$ and $p(1), \ldots, p(r) \in \mathbf{Z}$ such that

$$\left| \prod_{k=1}^{N} g_k(x_j) - \prod_{i=1}^{r} \{g_{k(i)}(x_j)\}^{p(i)} \right| < \varepsilon.$$

There is no loss of generality in assuming that $k(i) = i$ for all $i = 1, \ldots, r$. Define $h = (g_1 \cdots g_N)/(g_1^{p(1)} \cdots g_r^{p(r)})$. Then h in an element of $\Gamma(\nu)$, $h = 1$ on F_{j-1}, $|h(x_j) - 1| < \varepsilon$, and

$$\int |h - f| \, d\rho = \sum_{k=1}^{r} \int_{E_k} |h - f| \, d\rho \leqslant 2 \sum_{k=1}^{r} \rho(E_k) < \varepsilon.$$

Consequently we have proved that there exists a sequence (h_m) in $\Gamma(\nu)$ such that $|h_m - 1| < 1/m$ on F_j and $\int |h_m - f| \, d\rho < 1/m$ for all $m \geqslant 1$. Noting that F is a finite set and passing to a subsequence, we may assume that (h_m) converges to an element $g \in \Gamma(\nu)$. It is obvious that $g = 1$ on F_j and $g = f$ a.e. $d\rho$, which establishes our induction.

Finally we apply the above result for $j = n$ and $f = \gamma_1$. Thus there exists $g \in \Gamma(\nu)$ such that $g = \gamma_1$ a.e. $d\rho$ and $g = 1$ on $F_n = F$. Since Γ is dense in $\Gamma(\nu)$, this completes the proof.

PROOF OF THE THEOREM. Let $\mu, \mu_1, \ldots, \mu_n \in M(G)$ and U be as in the hypotheses of the Theorem. Let $b(\Gamma)$ denote the Bohr compactification of Γ and let $\chi \in b(\Gamma)$ be given. We must prove that χ belongs to the closure of $\{\gamma \in \Gamma: (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$ in $b(\Gamma)$.

To this end, choose any finite subset F of G and any $\eta > 0$. Then there exists γ_0 in Γ such that $|\chi - \gamma_0| < \eta/2$ on F. Let $\gamma_1 \in \Gamma$ and $\varepsilon > 0$ be such that

$$\{(\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)): \gamma \in \Gamma, |\hat{\mu}_j(\gamma_1) - \hat{\mu}_j(\gamma)| < \varepsilon\ \forall j\} \subset U.$$

Now we define $\rho = |\mu_1| + \cdots + |\mu_n| \in M(G)$. Then ρ is a nonnegative continuous measure and has property (HK), as is easily seen. It follows from Lemma 2 that there exists γ_2 in Γ such that $|\gamma_0 - \gamma_2| < \eta/2$ on F and $\int|\gamma_1 - \gamma_2|\, d\rho < \varepsilon$. Then we have $|\chi - \gamma_2| \leqslant |\chi - \gamma_0| + |\gamma_0 - \gamma_2| < \eta$ on F, and

$$|\hat{\mu}_j(\gamma_1) - \hat{\mu}_j(\gamma_2)| \leqslant \int|\gamma_1 - \gamma_2|\, d|\mu_j| \leqslant \int|\gamma_1 - \gamma_2|\, d\rho < \varepsilon$$

for all $j = 1, 2, \ldots, n$. Therefore $(\hat{\mu}_1(\gamma_2), \ldots, \hat{\mu}_n(\gamma_2))$ is in U. Since F was an arbitrary finite set in G and $\eta > 0$ was arbitrary, this implies that χ is in the closure of the set $\{\gamma \in \Gamma: (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$. The proof is complete.

In order to state a corollary to the Theorem, we let $q(G)$ denote the largest member of $\{2, 3, \ldots, \infty\}$ such that every neighborhood of $0 \in G$ contains an element of order q. Let $D(G) = \{z \in C: |z| \leqslant 1\}$ if $q(G) = \infty$, and let $D(G)$ be the convex hull of $\{\exp(2\pi i k/q(G)): k \in \mathbf{Z}\}$ in the complex plane if $q(G) < \infty$.

COROLLARY. *There exists a family* $\{\mu_t: 0 \leqslant t \leqslant 1\}$ *of continuous probability measures in* $M(G)$ *such that whenever* U *is a nonempty open subset of* $D(G)^{[0,1]}$, *then* $\{\gamma \in \Gamma: (\hat{\mu}_t(\gamma))_t \in U\}$ *is dense in the Bohr compactification of* Γ.

PROOF. As is well known (cf. [1]), G contains a compact perfect set K which is either a Kronecker set (if $q(G) = \infty$) or a $K_{q(G)}$-set (if $q(G) < \infty$). Let $\{E_t: 0 \leqslant t \leqslant 1\}$ be any family of pairwise disjoint perfect subsets of K, and let μ_t be any continuous probability measure concentrated on E_t $(0 \leqslant t \leqslant 1)$. Then it is easy to show that $\{(\hat{\mu}_t(\gamma))_t: \gamma \in \Gamma\}$ is dense in $D(G)^{[0,1]}$. Therefore the required result is an immediate consequence of the present theorem.

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