BOHR COMPACTIFICATION AND CONTINUOUS MEASURES

SADAHIRO SAEKI

Abstract. Let $G$ be an LCA group with dual $\Gamma$. As a consequence of our main result, it is shown that every continuous regular measure $\mu$ concentrated on a Kronecker set and with norm $> 1$ has the property that $\{|\hat{\mu}| > 1\}$ is dense in the Bohr compactification of $\Gamma$.

Y. Katznelson [3] constructs a continuous measure $\mu$ on the circle group such that $\{n \in \mathbb{Z} : |\hat{\mu}(n)| > 1\}$ is dense in the Bohr compactification of $\mathbb{Z}$. In the present note we point out that every continuous measure (with norm $> 1$) concentrated on a Kronecker set has this property.

Let $G$ be a nondiscrete LCA group with dual $\Gamma$, and let $M(G)$ be the convolution measure algebra of $G$ (cf. [2]). For $\mu \in M(G)$, we denote by $\hat{\mu}$ the Fourier-Stieltjes transform of $\mu$. It is easy to show that every measure $\mu \in M(G)$ concentrated on a Kronecker set (or a $K_p$-set) has the following property, which we call (HK): Given $\gamma \in \Gamma$, a Borel set $E$ in $G$, and $\varepsilon > 0$, there exists $\chi \in \Gamma$ such that

$$\int_E |\gamma - \chi| \, d|\mu| + \int_{G \setminus E} |1 - \chi| \, d|\mu| < \varepsilon.$$

Theorem. Let $\mu \in M(G)$ be a continuous measure having property (HK), and let $\mu_1, \ldots, \mu_n \in M(G)$ be absolutely continuous with respect to $\mu$. Then, for each nonempty (relatively) open subset $U$ of $\{(\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) : \gamma \in \Gamma\} \subset \mathbb{C}^n$, the set

$$\{\gamma \in \Gamma : (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$$

is dense in the Bohr compactification of $\Gamma$.

To prove this, we need two lemmas. Let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the circle group.

Lemma 1. Given $\varepsilon > 0$, there exists a natural number $r$ having the following property. If $z_1, z_2, \ldots, z_N$ are finitely many elements of $T$, there exist $k(1), k(2), \ldots, k(r) \in \{1, 2, \ldots, N\}$ and $p(1), p(2), \ldots, p(r) \in \mathbb{Z}$ such that

$$|z_1 \cdots z_N - z_{k(1)}^{p(1)} \cdots z_{k(r)}^{p(r)}| < \varepsilon.$$

Proof. Let $S$ be the subgroup of $T$ that is generated by all elements of $T$ having order less than $2\pi/\varepsilon$. Then $S$ is a finite group. We define $r$ to be the order of $S$.

Received by the editors September 26, 1979.


Key words and phrases. Bohr compactification, continuous measure, Kronecker set, $K_p$-set.

© 1980 American Mathematical Society

0002-9939/80/0000-0510/$01.75

244
Now let \( z_1, \ldots, z_N \in T \) be given. If \( \text{ord}(z_j) > 2\pi/e \) for some index \( j \), then \( \{z_k^p : p \in \mathbb{Z}\} \) is \( \varepsilon \)-dense in \( T \). Therefore it suffices to set \( k(1) = k(2) = \cdots = k(r) = j \), \( p(1) = p \) for some \( p \in \mathbb{Z} \), and \( p(2) = \cdots = p(r) = 0 \). So assume that \( \text{ord}(z_k) < 2\pi/e \) for all indices \( k \). Then \( \{z_k : 1 < k < N\} \) is contained in \( S \) and therefore consists of at most \( r \) different elements. Evidently this completes the proof.

**Lemma 2.** Suppose \( \rho \in M(G) \) is a nonnegative continuous measure having property (HK), \( F \) is a finite set in \( G \), \( \gamma_0, \gamma_1 \in \Gamma \) and \( \varepsilon > 0 \). Then there exists \( \gamma_2 \in \Gamma \) such that

\[
|\gamma_0 - \gamma_2| < \varepsilon \quad \text{on } F \quad \text{and} \quad \int |\gamma_1 - \gamma_2| \, d\rho < \varepsilon.
\]

**Proof.** Replacing \( \gamma_1 \) by \( \tilde{\gamma}_0 \gamma_1 \), we may assume that \( \gamma_0 = 1 \). Let us enumerate the elements of \( F \) as \( x_1, \ldots, x_n \), and define \( F_0 = \emptyset \) and \( F_j = \{x_1, \ldots, x_j\} \) for \( j = 1, 2, \ldots, n \). Put \( \nu = \delta_1 + \cdots + \delta_n + \rho \), where \( \delta_j \) is the unit point measure at \( x_j \). For \( \mu \in M(G) \), we denote by \( \Gamma(\mu) \) the closure of \( \Gamma \) in \( L^1(\mu) \). Notice that \( \Gamma(\mu) \) forms a multiplicative group.

By induction on \( j = 0, 1, \ldots, n \), we shall prove that given \( f \in \Gamma(\rho) \), there exists \( g \in \Gamma(\nu) \) such that \( g = f \) a.e. \( dp \) and \( g = 1 \) on \( F_j \). Since \( F \) is a finite set and \( F_0 = \emptyset \), this is obvious for \( j = 0 \). So assume that \( 1 < j < n \) and that the result is true for \( j - 1 \). Choose and fix any \( f \in \Gamma(\rho) \) and any \( \varepsilon > 0 \).

Let \( r = r(\varepsilon) \) be the natural number given by Lemma 1. Since \( \rho \) is a continuous measure, we can partition \( \Gamma \) into disjoint Borel sets \( E_1, E_2, \ldots, E_N \) such that \( \rho(E_k) < \varepsilon/(2r) \) for all \( k = 1, 2, \ldots, N \). Without loss of generality, assume that \( N > r \). Since \( \rho \) has property (HK) and \( f \in \Gamma(\rho) \), it is obvious that there exist \( f_1, f_2, \ldots, f_N \) in \( \Gamma(\rho) \) such that \( f_k = f \) a.e. \( dp \) on \( E_k \), and \( f_k = 1 \) a.e. \( dp \) on \( G \setminus E_k \). It follows from the inductive hypothesis that there exist \( g_1, g_2, \ldots, g_N \) in \( \Gamma(\nu) \) such that \( g_k = f \) a.e. \( dp \) on \( E_k \), \( g_k = 1 \) a.e. \( dp \) on \( G \setminus E_k \), and \( g_k = 1 \) on \( F_{j-1} \). We now apply Lemma 1 with \( z_k = g_k(x_j) \) to find \( k(1), \ldots, k(r) \in \{1, 2, \ldots, N\} \) and \( p(1), \ldots, p(r) \in \mathbb{Z} \) such that

\[
\left| \prod_{k=1}^{N} g_k(x_j) - \prod_{i=1}^{r} \{ g_{k(i)}(x_j) \}^{p(i)} \right| < \varepsilon.
\]

There is no loss of generality in assuming that \( k(i) = i \) for all \( i = 1, \ldots, r \). Define \( h = (g_1 \cdots g_N)/(g_1^{p(1)} \cdots g_N^{p(r)}) \). Then \( h \) in an element of \( \Gamma(\nu) \), \( h = 1 \) on \( F_{j-1} \), \( |h(x_j) - 1| < \varepsilon \), and

\[
\int |h - f| \, d\rho = \sum_{k=1}^{r} \int_{E_k} |h - f| \, d\rho < 2 \sum_{k=1}^{r} \rho(E_k) < \varepsilon.
\]

Consequently we have proved that there exists a sequence \( (h_m) \) in \( \Gamma(\nu) \) such that \( |h_m - 1| < 1/m \) on \( F_j \) and \( \int |h_m - f| \, d\rho < 1/m \) for all \( m > 1 \). Noting that \( F \) is a finite set and passing to a subsequence, we may assume that \( (h_m) \) converges to an element \( g \in \Gamma(\nu) \). It is obvious that \( g = 1 \) on \( F_j \) and \( g = f \) a.e. \( dp \), which establishes our induction.

Finally we apply the above result for \( j = n \) and \( f = \gamma_1 \). Thus there exists \( g \in \Gamma(\nu) \) such that \( g = \gamma_1 \) a.e. \( dp \) and \( g = 1 \) on \( F_n = F \). Since \( \Gamma \) is dense in \( \Gamma(\nu) \), this completes the proof.
Proof of the Theorem. Let $\mu, \mu_1, \ldots, \mu_n \in M(G)$ and $U$ be as in the hypotheses of the Theorem. Let $b(\Gamma)$ denote the Bohr compactification of $\Gamma$ and let $\chi \in b(\Gamma)$ be given. We must prove that $\chi$ belongs to the closure of $\{\gamma \in \Gamma: (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$ in $b(\Gamma)$.

To this end, choose any finite subset $F$ of $G$ and any $\eta > 0$. Then there exists $\gamma_0$ in $\Gamma$ such that $|\chi - \gamma_0| < \eta/2$ on $F$. Let $\gamma_1 \in \Gamma$ and $\epsilon > 0$ be such that

$$\left\{ (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) : \gamma \in \Gamma, |\hat{\mu}_j(\gamma_1) - \hat{\mu}_j(\gamma)| < \epsilon \forall j \right\} \subset U.$$ 

Now we define $\rho = |\mu_1| + \cdots + |\mu_n| \in M(G)$. Then $\rho$ is a nonnegative continuous measure and has property (HK), as is easily seen. It follows from Lemma 2 that there exists $\gamma_2$ in $\Gamma$ such that $|\gamma_0 - \gamma_2| < \eta/2$ on $F$ and $\int |\gamma_1 - \gamma_2| d\rho < \epsilon$. Then we have $|\chi - \gamma_2| < |\chi - \gamma_0| + |\gamma_0 - \gamma_2| < \eta$ on $F$, and

$$|\hat{\mu}_j(\gamma_1) - \hat{\mu}_j(\gamma_2)| \leq \int |\gamma_1 - \gamma_2| d|\mu_j| < \int |\gamma_1 - \gamma_2| d\rho < \epsilon$$

for all $j = 1, 2, \ldots, n$. Therefore $(\hat{\mu}_1(\gamma_2), \ldots, \hat{\mu}_n(\gamma_2))$ is in $U$. Since $F$ was an arbitrary finite set in $G$ and $\eta > 0$ was arbitrary, this implies that $\chi$ is in the closure of the set $\{\gamma \in \Gamma: (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$. The proof is complete.

In order to state a corollary to the Theorem, we let $q(G)$ denote the largest member of $\{2, 3, \ldots, \infty\}$ such that every neighborhood of $0 \in G$ contains an element of order $q$. Let $D(G) = \{z \in \mathbb{C}: |z| < 1\}$ if $q(G) = \infty$, and let $D(G)$ be the convex hull of $\{\exp(2\pi ik/q(G)): k \in \mathbb{Z}\}$ in the complex plane if $q(G) < \infty$.

Corollary. There exists a family $\{\mu_t: 0 < t < 1\}$ of continuous probability measures in $M(G)$ such that whenever $U$ is a nonempty open subset of $D(G)^{[0,1]}$, then $\{\gamma \in \Gamma: (\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma)) \in U\}$ is dense in the Bohr compactification of $\Gamma$.

Proof. As is well known (cf. [1]), $G$ contains a compact perfect set $K$ which is either a Kronecker set (if $q(G) = \infty$) or a $K_{(q(G))}$-set (if $q(G) < \infty$). Let $\{E_t: 0 < t < 1\}$ be any family of pairwise disjoint perfect subsets of $K$, and let $\mu_t$ be any continuous probability measure concentrated on $E_t (0 < t < 1)$. Then it is easy to show that $(\hat{\mu}_1(\gamma), \ldots, \hat{\mu}_n(\gamma))$ is dense in $D(G)^{[0,1]}$. Therefore the required result is an immediate consequence of the present theorem.

References


Department of Mathematics, Tokyo Metropolitan University, Fukazawa, Setagaya, Tokyo 158, Japan