EIGENVALUE DISTRIBUTION OF TOEPLITZ MATRICES

PAUL G. NEVAI


1. Let $d\alpha$ be a positive measure on the real line. Suppose that all the moments of $d\alpha$ are finite and the support of $d\alpha$ is an infinite set. Then there exist a sequence of polynomials $p_n(\alpha, x) = \gamma_n x^n + \ldots$, $n = 0, 1, 2, \ldots$, $\gamma_n > 0$, such that

$$\int_{-\infty}^{\infty} p_k(\alpha, x)p_l(\alpha, x) d\alpha(x) = \delta_{kl}.$$ 

If $g$ is a real valued $d\alpha$-measurable function and all the moments of $g d\alpha$ are finite then we can form a matrix $T(g, \alpha) = (a_{kl})_{k,l=0}^{n}$ defined by

$$a_{kl} = \int_{-\infty}^{\infty} p_k(\alpha, x)p_l(\alpha, x)g(x) d\alpha(x).$$

Such a matrix $T(g, \alpha)$ is called Toeplitz matrix corresponding to $d\alpha$ and generated by $g$. For $n = 1, 2, \ldots$ the truncated matrix $T_n(g, \alpha)$ is defined by $T_n(g, \alpha) = (a_{kl})_{k,l=0}^{n-1}$. Since $T_n(g, \alpha)$ is Hermitian, its eigenvalues $\Lambda_{kn}(g, \alpha)$, $k = 1, 2, \ldots, n$, are all real.

The purpose of this paper is to investigate the behavior of $\Lambda_{kn}$ as $n \to \infty$. This problem is a very old one and it has many applications in mathematics and physics. Let us just mention the book [3] of U. Grenander and G. Szegö that is devoted to such questions. In particular, Theorem 7.7(b) of [3] states that if $\alpha$ is in the Szegö class, that is supp$(\alpha) = [-1,1]$ and log $\alpha'(\cos \theta) \in L^1$, and $g$ is continuous in $[-1, 1]$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} G(\Lambda_{kn}(g, \alpha)) = \frac{1}{\pi} \int_{-1}^{1} \frac{G(x)}{(1-x^2)^{1/2}} dx$$

whenever $G$ is continuous. A correct proof of a somewhat more general form of this theorem was given in [4, p. 55]. In this paper I will show that the condition on $g$ can be relaxed and the same conclusion holds if $g$ only belongs to $L^\infty$. Note that $C$ is not dense in $L^\infty$ so that continuity arguments cannot be used in the process of generalization. More precisely, I will have to introduce a new method of investigating distribution of eigenvalues of Toeplitz matrices. The following result is basically
a combination of the condition of Theorem 5.2(b) and the conclusion of Theorem 7.7(b) of [3].

**Theorem 1.** Let supp(\(\sigma_\alpha\)) = \([-1, 1]\) and \(\alpha'(x) > 0\) for almost every \(x \in [-1, 1]\). Assume that the Toeplitz matrix \(T(g, \sigma_\alpha)\) is generated by an \(L^\infty\) function \(g\). Let \(G\) be a continuous function in an interval containing the essential range of \(g\). Then the eigenvalues \(\Lambda_{kn}\) of the truncated matrix \(T_n(g, \sigma_\alpha)\) satisfy

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} G(\Lambda_{kn}) = \frac{1}{\pi} \int_{-1}^{1} \frac{G(g(x))}{(1 - x^2)^{1/2}} \, dx.
\]

Note that the proof of this theorem does not require the use of Erdös-Turan's celebrated result [1, p. 547] on the distribution of zeros of orthogonal polynomials. Actually, the case \(g(x) = x\) of Theorem 1 is equivalent to the theorem of Erdös and Turan. I think that the proof of Theorem 1 is more elementary than any known proof of the Erdös-Turan theorem. Theorem 1 is an easy consequence of the following result which is likely to have other applications as well. The function \(K_n(\sigma_\alpha, x, t)\) below is defined by

\[
K_n(\sigma_\alpha, x, t) = \sum_{k=0}^{n-1} p_k(\sigma_\alpha, x)p_k(\sigma_\alpha, t).
\]

**Theorem 2.** Let \(\sigma_\alpha\) be such that supp(\(\sigma_\alpha\)) = \([-1, 1]\) and \(\alpha'(x) > 0\) almost everywhere in \([-1, 1]\). Suppose that \(f\) belongs to \(L^\infty\) in the square \([-1, 1] \times [1, 1]\) and satisfies

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{x}^{x+\varepsilon} |f(x, t) - f(x, x)| \, dt = 0
\]

for almost every \(x \in (-1, 1)\). Then

\[
\lim_{n \to \infty} \frac{1}{n} \int_{-1}^{1} K_n^2(\sigma_\alpha, x, t)f(x, t) \, d\alpha(x) \, d\alpha(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x, x)}{(1 - x^2)^{1/2}} \, dx.
\]

I do not know of any result which would help to find out whether a given Toeplitz matrix \(T(g, \sigma_\alpha)\) is generated by a continuous function \(g\). On the other hand, Theorem 7.7(b) of [3] assumes continuity of \(g\). The following result justifies Theorem 1 since it decides whether \(T(g, \sigma_\alpha)\) is generated by an \(L^\infty\) function \(g\).

**Theorem 3.** Let \(T(g, \sigma_\alpha)\) be a Toeplitz matrix corresponding to \(\sigma_\alpha\) supported in \([-1, 1]\) and satisfying \(\alpha' > 0\) almost everywhere in \([-1, 1]\). Let \(m\) and \(M\) be defined by

\[
m = \inf_{n,k} \{\Lambda_{kn}(g, \sigma_\alpha)\}, \quad M = \sup_{n,k} \{\Lambda_{kn}(g, \sigma_\alpha)\}.
\]

Then the essential range of \(g\) lies between \(m\) and \(M\). In particular, if both \(m\) and \(M\) are finite then \(g\) is an \(L^\infty\) function. Furthermore, if

\[
\sup_{t} \sum_{k=0}^{\infty} \left| \int_{-1}^{1} p_k(\sigma_\alpha, x)p_k(\sigma_\alpha, x)g(x) \, d\alpha(x) \right| < \infty
\]

then \(g \in L^\infty\).
It might be interesting to note that the converse statement
\[ \text{ess inf } g \leq m < M \leq \text{ess sup } g \]
is well known and trivial.

2. This section is devoted to the proof of Theorems 1–3. Besides (1) the following notation will be used:
\[ \lambda_n(\alpha, x) = K_n^{-1}(\alpha, x, x). \]

**Proof of Theorem 2.** Let \( dv \) denote the Chebyshev measure of \([-1, 1]\), that is, let \( \text{supp}(dv) = [-1, 1] \) and \( v(x) = \int_{-1}^{1} dt/(1 - t^2)^{1/2} \) for \( x \in [-1, 1] \). For \( i = 1, 2 \) define \( g_i \) by
\[ g_i(x, t) = 1 + \|f\|_{\infty} + (-1)^i f(x, t). \quad (3) \]

Our first goal is to prove
\[ \lim_{n \to \infty} \int_{-1}^{1} K_n^2(dv, x, t) g_i(x, t)^{-1} \alpha(t) dt = (1 - x^2)^{1/2} \alpha'(x) g_i(x, x)^{-1} \quad (4) \]
for almost every \( x \in [-1, 1] \). Let \( d\beta \) denote the sum of the singular and jump components of \( d\alpha \). By Lemma 6.2.31 of [4, p. 92]
\[ \lim_{n \to \infty} \int_{-1}^{1} K_n^2(dv, x, t) d\beta(t) = 0 \]
almost everywhere in \([-1, 1]\). Since by (3) \( 0 \leq g_i^{-1} \leq 1 \), this implies
\[ \lim_{n \to \infty} \int_{-1}^{1} K_n^2(dv, x, t) g_i(x, t)^{-1} d\beta(t) = 0 \]
for almost every \( x \in [-1, 1] \). Thus to prove (4) we need to show
\[ \lim_{n \to \infty} \int_{-1}^{1} K_n^2(dv, x, t) g_i(x, t)^{-1} \alpha'(t) dt = (1 - x^2)^{1/2} \alpha'(x) g_i(x, x)^{-1} \quad (5) \]
almost everywhere in \([-1, 1]\). Let \( h_i \) be defined by
\[ h_i(x, t) = g_i(x, t)^{-1} \alpha'(t)(1 - t^2)^{1/2}. \quad (6) \]
Since \( \lambda_n(dv, x) \int_{-1}^{1} K_n^2(dv, x, t) dt/(1 - t^2)^{1/2} = 1 \), formula (5) is equivalent to
\[ \lim_{n \to \infty} \int_{-1}^{1} K_n^2(dv, x, t) [h_i(x, t) - h_i(x, x)] dt/(1 - t^2)^{1/2} = 0. \quad (7) \]

Note that the integrability of \( \alpha' \) and formulas (2), (3) and (6) imply
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{x}^{x+\varepsilon} |h_i(x, t) - h_i(x, x)| dt = 0 \quad (8) \]
for almost every \( x \in (-1, 1) \). Now fix \( x \) for which (8) holds and choose \( \delta > 0 \) so that \(-1 < x - \delta < x + \delta < 1\). Using the well-known estimate
\[ \lambda_n(dv, x) K_n^2(dv, x, t) < 3 \min\{n, (n(x - t)^{-2}\} \quad [2, pp. 106–108] \]
we obtain
\[
\left| \lambda_n (dv, x) \int_{-1}^{1} K_n^2 (dv, x, t) \left[ h_i (x, t) - h_i (x, x) \right] \frac{dt}{(1 - t^2)^{1/2}} \right| < 3 n \int_{x-1/n}^{x+1/n} \left| h_i (x, t) - h_i (x, x) \right| \frac{dt}{(1 - t^2)^{1/2}} + 3 \int_{1/n < |x-t| < \delta} \frac{\left| h_i (x, t) - h_i (x, x) \right|}{(x-t)^2} \frac{dt}{(1 - t^2)^{1/2}} + \frac{3}{n^8^2} \int_{|x-t| > \delta; 0 < t < 1} \left[ |h_i (x, t)| + |h_i (x, x)| \right] \frac{dt}{(1 - t^2)^{1/2}} \equiv I_1 + I_2 + I_3
\]

provided that \( 1/n < \delta \). Applying (8) we get \( \lim I_1 = 0 \) as \( n \to \infty \). Formula (6) implies that \( I_3 \) also tends to 0 as \( n \to \infty \). \( I_2 \) can be estimated by using Lebesgue's method of integration by parts. Denoting by \( C \) the maximum of \( (1 - t^2)^{-1/2} \) on \([x - \delta, x + \delta]\) we obtain

\[
I_3 < 3C \left\{ n \int_{x-1/n}^{x+1/n} |h_i (x, u) - h_i (x, x)| du + \frac{1}{n^8^2} \int_{x-\delta}^{x+2\delta} |h_i (x, u) - h_i (x, x)| du + \frac{1}{n} \int_{1/n < |x-t| < \delta} \int_{x}^{t} |h_i (x, u) - h_i (x, x)| du dt \right\}.
\]

Thus by (8)

\[
\limsup_{n \to \infty} I_3 < 6C \sup_{|e| < \delta} \left| \frac{1}{e} \int_{x}^{x+\epsilon} |h_i (x, u) - h_i (x, x)| du \right|.
\]

Therefore we get the inequality

\[
\limsup_{n \to \infty} \left| \lambda_n (dv, x) \int_{-1}^{1} K_n^2 (dv, x, t) \left[ h_i (x, t) - h_i (x, x) \right] \frac{dt}{(1 - t^2)^{1/2}} \right| < 6C \sup_{|e| < \delta} \left| \frac{1}{e} \int_{x}^{x+\epsilon} |h_i (x, u) - h_i (x, x)| du \right|.
\]

Now letting \( \delta \to 0 \) and using (8) we obtain (7). Having established (4) the theorem can easily be proved. Since \( \alpha' > 0 \) almost everywhere in \([-1, 1]\) and \( \lim_{n \to \infty} (1/n)K_n (dv, x, x) = 1/\pi \) for \(-1 < x < 1\) [4, p. 79], formula (4) implies

\[
\lim_{n \to \infty} \frac{1}{n} \int_{-1}^{1} K_n^2 (dv, x, x) \alpha'(x) dx \int_{-1}^{1} K_n^2 (dv, x, t) g_i (x, x) \frac{dt}{(1 - x^2)^{1/2}} = \frac{1}{\pi} \frac{g_i (x, x)}{(1 - x^2)^{1/2}}
\]

for almost every \( x \in (-1, 1) \). Thus by Fatou's lemma

\[
\liminf_{n \to \infty} \frac{1}{n} \int_{-1}^{1} K_n^2 (dv, x, x) \alpha'(x) \frac{dx}{(1 - x^2)^{1/2}} = \frac{1}{\pi} \int_{-1}^{1} g_i (x, x) \frac{dx}{(1 - x^2)^{1/2}}.
\]
The next step is to use the identity
\[ K^2_n((dv, x, x)) = \int_{-1}^{1} K_n((dv, x, t))K_n((da, x, t)) \, da(t) \]
and Schwarz's inequality to obtain

\[ K^2_n((dv, x, x)) \leq \int_{-1}^{1} \int_{-1}^{1} K^2_n((da, x, t))g_i(x, t) \, da(t) \, da(x) \]

This inequality allows us to conclude that

\[ \int_{-1}^{1} \int_{-1}^{1} \frac{K^2_n((da, x, x)) \alpha'(x)}{K^2_n((dv, x, x))g_i(x, t)} \, da(t) \]

which together with (9) leads to

\[ \lim_{n \to \infty} \frac{1}{n} \int_{-1}^{1} \int_{-1}^{1} K^2_n((da, x, t))g_i(x, t) \, da(x) \, da(t) \]

Now applying the definition of \( g_i \) (3) and the identities

\[ \frac{1}{\pi} \int_{-1}^{1} \frac{dx}{(1 - x^2)^{1/2}} = \frac{1}{n} \int_{-1}^{1} \int_{-1}^{1} K^2_n((da, x, t)) \, da(x) \, da(t) = 1 \]

for \( n = 1, 2, \ldots \), we get

\[ \lim_{n \to \infty} \frac{1}{n} \int_{-1}^{1} \int_{-1}^{1} K^2_n((da, x, t))f(x, t) \, da(x) \, da(t) \]

for \( i = 1, 2 \). The last inequality is obviously equivalent to the statement of the theorem.

**Lemma 4.** Let \( T_n((g, da)) \) be the truncated matrix of an arbitrary Toeplitz matrix \( T((g, da)) \). Then there exist \( n \) polynomials \( \varphi_{kn}(g, da, x) \) of degree at most \( n - 1 \) such that

\[ \int_{-\infty}^{\infty} \varphi_{kn}(g, da, x) \varphi_{ln}(g, da, x) \, da(x) = \delta_{kl}, \]

\[ \int_{-\infty}^{\infty} \varphi_{kn}(g, da, x) \varphi_{ln}(g, da, x)g(x) \, da(x) = \Lambda_{kn}(g, da) \delta_{kl} \]

and

\[ \sum_{k=1}^{n} \varphi_{kn}(g, da, x) \varphi_{kn}(g, da, t) = K_n((da, x, t)). \]

**Proof.** Let \( b_k = (b^0_k, b^1_k, \ldots, b^{n-1}_k), k = 1, 2, \ldots, n, \) be a collection of orthonormalized eigenvectors of \( T_n((g, da)) \), that is \( T_n b_k = \Lambda_{kn} b_k \) and \( (b_k, b_l) = \delta_{kl} \).

Define \( \varphi_{kn}(x) \) by \( \varphi_{kn}(x) = \sum_{j=0}^{n-1} b^j_k p_j((da, x)) \). Then

\[ \int_{-\infty}^{\infty} \varphi_{kn}(x) \varphi_{ln}(x) \, da(x) = (b_k, b_l) = \delta_{kl} \]
Since the number of the polynomials \( \varphi_{kn} \) is exactly \( n \), the orthogonality relation (11) guarantees that \( \sum_{k=1}^{n} \varphi_{kn}(x) \varphi_{kn}(t) \) is a reproduction kernel for the space of polynomials of degree at most \( n - 1 \) supplied with the inner product of \( L^2(\alpha) \). Now (10) follows from the uniqueness of the reproduction kernel.

**Proof of Theorem 1.** Note that \( f(x, t) = G(g(t)) \) obviously satisfies (2) of Theorem 2. Thus, since

\[
\int_{-1}^{1} G(g(x)) K_n(\alpha, x, x) \, d\alpha(x) = \int_{-1}^{1} G(g(t)) K_n^2(\alpha, x, t) \, d\alpha(x) \, d\alpha(t),
\]

the theorem will be proved if it can show the validity of

\[
\lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{k=1}^{n} G(\Lambda_{kn}) - \frac{1}{n} \int_{-1}^{1} G(g(x)) K_n(\alpha, x, x) \, d\alpha(x) \right\} = 0. \tag{12}
\]

By Lemma 4 we have

\[
\frac{1}{n} \sum_{k=1}^{n} G(\Lambda_{kn}) - \frac{1}{n} \int_{-1}^{1} G(g(x)) K_n(\alpha, x, x) \, d\alpha(x) = \frac{1}{n} \sum_{k=1}^{n} \int_{|\Lambda_{kn} - g(x)| < \delta} \left[ G(\Lambda_{kn}) - G(g(x)) \right] \varphi_{kn}^2(x) \, d\alpha(x) \equiv I
\]

where \( \varphi_{kn}(x) = \varphi_{kn}(g, \alpha, x) \). Now fix \( \epsilon > 0 \) and choose \( \delta > 0 \) so that \( |G(x) - G(y)| < \epsilon \) for \( |x - y| < \delta \). Then we can write

\[
I = \frac{1}{n} \sum_{k=1}^{n} \int_{|\Lambda_{kn} - g(x)| < \delta} \left[ G(\Lambda_{kn}) - G(g(x)) \right] \varphi_{kn}^2(x) \, d\alpha(x)
\]  

\[
+ \frac{1}{n} \sum_{k=1}^{n} \int_{|\Lambda_{kn} - g(x)| > \delta} \left[ G(\Lambda_{kn}) - G(g(x)) \right] \varphi_{kn}^2(x) \, d\alpha(x) \equiv I^1 + I^2.
\]

By the choice of \( \delta \)

\[
|I^1| < \epsilon \frac{1}{n} \sum_{k=1}^{n} \int_{-1}^{1} \varphi_{kn}^2(x) \, d\alpha(x) = \epsilon. \tag{13}
\]

We also have

\[
|I^2| < 2\delta^{-2} \max |G| \frac{1}{n} \sum_{k=1}^{n} \int_{-1}^{1} (g(x) - \Lambda_{kn})^2 \varphi_{kn}^2(x) \, d\alpha(x)
\]  

\[
= 2\delta^{-2} \max |G| \frac{1}{n} \sum_{k=1}^{n} \int_{-1}^{1} (g^2(x) - 2g(x)\Lambda_{kn} + \Lambda_{kn}^2) \varphi_{kn}^2(x) \, d\alpha(x).
\]

Using Lemma 4 we obtain

\[
|I^2| < 2\delta^{-2} \max |G| \frac{1}{n} \left\{ \int_{-1}^{1} g^2(x) K_n(\alpha, x, x) \, d\alpha(x) - \sum_{k=1}^{n} \Lambda_{kn}^2 \right\}. \tag{14}
\]

Since \( K_n(\alpha, x, x) = \int_{-1}^{1} K_n^2(\alpha, x, t) \, d\alpha(t) \) and

\[
\sum_{k=1}^{n} \Lambda_{kn}^2 = \text{Trace}(T_n^2(g, \alpha)) = \int_{-1}^{1} \int_{-1}^{1} g(x)g(t) K_n^2(\alpha, x, t) \, d\alpha(x) \, d\alpha(t),
\]
we can rewrite (14) in the form

$$|J^2| < 28^{-2} \max |G| \frac{1}{n} \int_{-1}^{1} \int_{-1}^{1} \left[ g^2(x) - g(x)g(t) \right] K_n^2(\alpha, x, t) \, d\alpha(x) \, d\alpha(t).$$

Note that the function $f(x, t) = g^2(x) - g(x)g(t)$ satisfies condition (2) of Theorem 2 and $f(x, x) = 0$ almost everywhere in $[-1, 1]$. Therefore $h_{mn} \to 0$ holds. Taking (13) into account we get $\limsup_{n \to \infty} I^2 = 0$. Since $\varepsilon$ is arbitrary, $I$ tends to 0 as $n \to \infty$; that is (12) has been proved.

**Proof of Theorem 3.** It is well known that for every polynomial $\pi$

$$\frac{\int_{-1}^{1} \pi^2(t)g(t) \, d\alpha(t)}{\int_{-1}^{1} \pi^2(t) \, d\alpha(t)} < M.$$  (15)

For $-1 < x < 1$ and $n = 1, 2, \ldots$, let $\pi(t)$ be defined by $\pi(t) = K_n(d\nu, x, t)$ where $d\nu$ denotes the Chebyshev measure of $[-1, 1]$. Then by (15) the inequality

$$\frac{\lambda_n(d\nu, x)\int_{-1}^{1} K_n^2(d\nu, x, t)g(t) \, d\alpha(t)}{\lambda_n(d\nu, x)\int_{-1}^{1} K_n^2(d\nu, x, t) \, d\alpha(t)} < M$$

holds. It follows from Lemma 6.2.32 of [4, p. 93] that the numerator and denominator in (16) converge to $(1 - x^2)^{1/2}g(x)\alpha'(x)$ and $(1 - x^2)^{1/2}\alpha'(x)$ respectively for almost every $x \in [-1, 1]$ as $n \to \infty$. Since $\alpha'(x) > 0$ almost everywhere in $[-1, 1]$, we obtain $g(x) < M$. Applying the same reasoning to $T(-g, d\alpha)$ we obtain $g(x) > m$ almost everywhere in $[-1, 1]$. The last part of the theorem follows from the first one and from Theorem 1.18(b) of [3].

**References**


**Department of Mathematics, Ohio State University, Columbus, Ohio 43220**