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EIGENVALUE DISTRIBUTION OF TOEPLITZ MATRICES

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ABSTRACT. A theorem of U. Grenander and G. Szegö on Toeplitz matrices is generalized. A new method is proposed for investigating eigenvalue distribution of Toeplitz matrices.

1. Let $d\alpha$ be a positive measure on the real line. Suppose that all the moments of $d\alpha$ are finite and the support of $d\alpha$ is an infinite set. Then there exist a sequence of polynomials $p_n(d\alpha, x) = \gamma_n x^n + \dots, n = 0, 1, 2, \dots, \gamma_n > 0$, such that

$$\int_{-\infty}^{\infty} p_k(d\alpha, x) p_l(d\alpha, x)\, d\alpha(x) = \delta_{kl}.$$

If g is a real valued $d\alpha$-measurable function and all the moments of $g d\alpha$ are finite then we can form a matrix $T(g, d\alpha) = (a_{kl})_{k,l=0}^{\infty}$ defined by

$$a_{kl} = \int_{-\infty}^{\infty} p_k(d\alpha, x) p_l(d\alpha, x) g(x)\, d\alpha(x).$$

Such a matrix $T(g, d\alpha)$ is called Toeplitz matrix corresponding to $d\alpha$ and generated by g. For $n = 1, 2, \dots$ the truncated matrix $T_n(g, d\alpha)$ is defined by $T_n(g, d\alpha) = (a_{kl})_{k,l=0}^{n-1}$. Since $T_n(g, d\alpha)$ is Hermitian, its eigenvalues $\Lambda_{kn}(g, d\alpha)$, $k = 1, 2, \dots, n$, are all real.

The purpose of this paper is to investigate the behavior of Λ_{kn} as $n \to \infty$. This problem is a very old one and it has many applications in mathematics and physics. Let us just mention the book [3] of U. Grenander and G. Szegö that is devoted to such questions. In particular, Theorem 7.7(b) of [3] states that if $d\alpha$ is in the Szegö class, that is $\operatorname{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos\theta) \in L^1$, and g is continuous in $[-1, 1]$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} G(\Lambda_{kn}(g, d\alpha)) = \frac{1}{\pi} \int_{-1}^{1} \frac{G(g(x))}{(1 - x^2)^{1/2}}\, dx$$

whenever G is continuous. A correct proof of a somewhat more general form of this theorem was given in [4, p. 55]. In this paper I will show that the condition on g can be relaxed and the same conclusion holds if g only belongs to L^∞. Note that C is not dense in L^∞ so that continuity arguments cannot be used in the process of generalization. More precisely, I will have to introduce a new method of investigating distribution of eigenvalues of Toeplitz matrices. The following result is basically

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a combination of the condition of Theorem 5.2(b) and the conclusion of Theorem 7.7(b) of [3].

THEOREM 1. *Let* supp$(d\alpha) = [-1, 1]$ *and* $\alpha'(x) > 0$ *for almost every* $x \in [-1, 1]$. *Assume that the Toeplitz matrix* $T(g, d\alpha)$ *is generated by an* L^∞ *function* g. *Let* G *be a continuous function in an interval containing the essential range of* g. *Then the eigenvalues* Λ_{kn} *of the truncated matrix* $T_n(g, d\alpha)$ *satisfy*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1} G(\Lambda_{kn}) = \frac{1}{\pi} \int_{-1}^{1} \frac{G(g(x))}{(1-x^2)^{1/2}}\, dx.$$

Note that the proof of this theorem does not require the use of Erdös-Turan's celebrated result [1, p. 547] on the distribution of zeros of orthogonal polynomials. Actually, the case $g(x) = x$ of Theorem 1 is equivalent to the theorem of Erdös and Turan. I think that the proof of Theorem 1 is more elementary than any known proof of the Erdös-Turan theorem. Theorem 1 is an easy consequence of the following result which is likely to have other applications as well. The function $K_n(d\alpha, x, t)$ below is defined by

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x) p_k(d\alpha, t). \tag{1}$$

THEOREM 2. *Let* $d\alpha$ *be such that* supp$(d\alpha) = [-1, 1]$ *and* $\alpha'(x) > 0$ *almost everywhere in* $[-1, 1]$. *Suppose that* f *belongs to* L^∞ *in the square* $[-1, 1] \times [1, 1]$ *and satisfies*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{x}^{x+\varepsilon} |f(x, t) - f(x, x)|\, dt = 0 \tag{2}$$

for almost every $x \in (-1, 1)$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} \int_{-1}^{1} K_n^2(d\alpha, x, t) f(x, t)\, d\alpha(x)\, d\alpha(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x, x)}{(1-x^2)^{1/2}}\, dx.$$

I do not know of any result which would help to find out whether a given Toeplitz matrix $T(g, d\alpha)$ is generated by a continuous function g. On the other hand, Theorem 7.7(b) of [3] assumes continuity of g. The following result justifies Theorem 1 since it decides whether $T(g, d\alpha)$ is generated by an L^∞ function g.

THEOREM 3. *Let* $T(g, d\alpha)$ *be a Toeplitz matrix corresponding to* $d\alpha$ *supported in* $[-1, 1]$ *and satisfying* $\alpha' > 0$ *almost everywhere in* $[-1, 1]$. *Let* m *and* M *be defined by*

$$m = \inf_{n,k} \{\Lambda_{kn}(g, d\alpha)\}, \quad M = \sup_{n,k} \{\Lambda_{kn}(g, d\alpha)\}.$$

Then the essential range of g *lies between* m *and* M. *In particular, if both* m *and* M *are finite then* g *is an* L^∞ *function. Furthermore, if*

$$\sup_{l} \sum_{k=0}^{\infty} \left| \int_{-1}^{1} p_k(d\alpha, x) p_l(d\alpha, x) g(x)\, d\alpha(x) \right| < \infty$$

then $g \in L^\infty$.

It might be interesting to note that the converse statement

$$\text{ess inf } g < m < M < \text{ess sup } g$$

is well known and trivial.

2. This section is devoted to the proof of Theorems 1–3. Besides (1) the following notation will be used:

$$\lambda_n(d\alpha, x) = K_n^{-1}(d\alpha, x, x).$$

PROOF OF THEOREM 2. Let dv denote the Chebyshev measure of $[-1, 1]$, that is, let $\operatorname{supp}(dv) = [-1, 1]$ and $v(x) = \int_{-1}^{x} dt/(1 - t^2)^{1/2}$ for $x \in [-1, 1]$. For $i = 1, 2$ define g_i by

$$g_i(x, t) = 1 + \|f\|_\infty + (-1)^i f(x, t). \tag{3}$$

Our first goal is to prove

$$\lim_{n\to\infty} \lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)g_i(x, t)^{-1}\, d\alpha(t) = (1 - x^2)^{1/2}\alpha'(x)g_i(x, x)^{-1} \tag{4}$$

for almost every $x \in [-1, 1]$. Let $d\beta$ denote the sum of the singular and jump components of $d\alpha$. By Lemma 6.2.31 of [4, p. 92]

$$\lim_{n\to\infty} \lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)\, d\beta(t) = 0$$

almost everywhere in $[-1, 1]$. Since by (3) $0 < g_i^{-1} < 1$, this implies

$$\lim_{n\to\infty} \lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)g_i(x, t)^{-1}\, d\beta(t) = 0$$

for almost every $x \in [-1, 1]$. Thus to prove (4) we need to show

$$\lim_{n\to\infty} \lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)g_i(x, t)^{-1}\, \alpha'(t)\, dt = (1 - x^2)^{1/2}\alpha'(x)g_i(x, x)^{-1} \tag{5}$$

almost everywhere in $[-1, 1]$. Let h_i be defined by

$$h_i(x, t) = g_i(x, t)^{-1}\alpha'(t)(1 - t^2)^{1/2}. \tag{6}$$

Since $\lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)\, dt/(1 - t^2)^{1/2} = 1$, formula (5) is equivalent to

$$\lim_{n\to\infty} \lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)[h_i(x, t) - h_i(x, x)]\, dt/(1 - t^2)^{1/2} = 0. \tag{7}$$

Note that the integrability of α' and formulas (2), (3) and (6) imply

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{x}^{x+\varepsilon} |h_i(x, t) - h_i(x, x)|\, dt = 0 \tag{8}$$

for almost every $x \in (-1, 1)$. Now fix x for which (8) holds and choose $\delta > 0$ so that $-1 < x - \delta < x + \delta < 1$. Using the well-known estimate $\lambda_n(dv, x)K_n^2(dv, x, t) \leq 3\min\{n, (n(x - t))^{-2}\}$ [2, pp. 106–108] we obtain

$$\left|\lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)[h_i(x, t) - h_i(x, x)] \frac{dt}{(1 - t^2)^{1/2}}\right|$$

$$< 3n\int_{x-1/n}^{x+1/n}|h_i(x, t) - h_i(x, x)| \frac{dt}{(1 - t^2)^{1/2}}$$

$$+ \frac{3}{n}\int_{1/n<|x-t|<\delta} \frac{|h_i(x, t) - h_i(x, x)|}{(x - t)^2} \frac{dt}{(1 - t^2)^{1/2}}$$

$$+ \frac{3}{n\delta^2}\int_{|x-t|>\delta;-1<t<1}[|h_i(x, t)| + |h_i(x, x)|]\frac{dt}{(1 - t^2)^{1/2}} \equiv I_1 + I_2 + I_3$$

provided that $1/n < \delta$. Applying (8) we get $\lim I_1 = 0$ as $n \to \infty$. Formula (6) implies that I_3 also tends to 0 as $n \to \infty$. I_2 can be estimated by using Lebesgue's method of integration by parts. Denoting by C the maximum of $(1 - t^2)^{-1/2}$ on $[x - \delta, x + \delta]$ we obtain

$$I_3 < 3C\left\{n\int_{x-1/n}^{x+1/n}|h_i(x, u) - h_i(x, x)|\, du\right.$$

$$+ \frac{1}{n\delta^2}\int_{x-\delta}^{x+\delta}|h_i(x, u) - h_i(x, x)|\, du$$

$$\left.+ \frac{1}{n}\int_{1/n<|x-t|<\delta}|x - t|^{-3}\left|\int_x^t|h_i(x, u) - h_i(x, x)|\, du\right| dt\right\}.$$

Thus by (8)

$$\limsup_{n\to\infty} I_3 < 6C \sup_{|\varepsilon|<\delta}\left|\frac{1}{\varepsilon}\int_x^{x+\varepsilon}|h_i(x, u) - h_i(x, x)|\, du\right|.$$

Therefore we get the inequality

$$\limsup_{n\to\infty}\left|\lambda_n(dv, x)\int_{-1}^{1} K_n^2(dv, x, t)[h_i(x, t) - h_i(x, x)] \frac{dt}{(1 - t^2)^{1/2}}\right|$$

$$< 6C \sup_{|\varepsilon|<\delta}\left|\frac{1}{\varepsilon}\int_x^{x+\varepsilon}|h_i(x, u) - h_i(x, x)|\, du\right|.$$

Now letting $\delta \to 0$ and using (8) we obtain (7). Having established (4) the theorem can easily be proved. Since $\alpha' > 0$ almost everywhere in $[-1, 1]$ and $\lim_{n\to\infty}(1/n)K_n(dv, x, x) = 1/\pi$ for $-1 < x < 1$ [4, p. 79], formula (4) implies

$$\lim_{n\to\infty} \frac{1}{n} \frac{K_n^2(dv, x, x)\alpha'(x)}{\int_{-1}^{1} K_n^2(dv, x, t)g_i(x, t)^{-1}\, d\alpha(t)} = \frac{1}{\pi} \frac{g_i(x, x)}{(1 - x^2)^{1/2}}$$

for almost every $x \in (-1, 1)$. Thus by Fatou's lemma

$$\liminf_{n\to\infty} \frac{1}{n}\int_{-1}^{1} \frac{K_n^2(dv, x, x)\alpha'(x)}{\int_{-1}^{1} K_n^2(dv, x, t)g_i(x, t)^{-1}\, d\alpha(t)}\, dx$$

$$> \frac{1}{\pi}\int_{-1}^{1} g_i(x, x) \frac{dx}{(1 - x^2)^{1/2}}. \tag{9}$$

The next step is to use the identity $K_n(dv, x, x) = \int_{-1}^{1} K_n(dv, x, t)K_n(d\alpha, x, t)\, d\alpha(t)$ and Schwarz's inequality to obtain

$$K_n^2(dv, x, x) \leq \int_{-1}^{1} K_n^2(dv, x, t)g_i^{-1}(x, t)\, d\alpha(t)\int_{-1}^{1} K_n^2(d\alpha, x, t)g_i(x, t)\, d\alpha(t).$$

This inequality allows us to conclude that

$$\int_{-1}^{1} \frac{K_n^2(d\alpha, x, x)\alpha'(x)}{\int_{-1}^{1} K_n^2(dv, x, t)g_i^{-1}(x, t)\, d\alpha(t)}\, dx \leq \int_{-1}^{1}\int_{-1}^{1} K_n^2(d\alpha, x, t)g_i(x, t)\, d\alpha(x)\, d\alpha(t)$$

which together with (9) leads to

$$\liminf_{n\to\infty} \frac{1}{n} \int_{-1}\int_{-1} K_n^2(d\alpha, x, t)g_i(x, t)\, d\alpha(x)\, d\alpha(t) \geq \frac{1}{\pi}\int_{-1}^{1} g_i(x, x)\, \frac{dx}{(1-x^2)^{1/2}}.$$

Now applying the definition of g_i (3) and the identities

$$\frac{1}{\pi}\int_{-1}^{1} \frac{dx}{(1-x^2)^{1/2}} = \frac{1}{n}\int_{-1}^{1}\int_{-1}^{1} K_n^2(d\alpha, x, t)\, d\alpha(x)\, d\alpha(t) = 1$$

for $n = 1, 2, \ldots,$ we get

$$\liminf_{n\to\infty} (-1)^i \frac{1}{n}\int_{-1}^{1}\int_{-1}^{1} K_n^2(d\alpha, x, t)f(x, t)\, d\alpha(x)\, d\alpha(t) \geq (-1)^i\frac{1}{\pi}\int_{-1}^{1} \frac{f(x, x)}{(1-x^2)^{1/2}}\, dx$$

for $i = 1, 2$. The last inequality is obviously equivalent to the statement of the theorem.

LEMMA 4. *Let $T_n(g, d\alpha)$ be the truncated matrix of an arbitrary Toeplitz matrix $T(g, d\alpha)$. Then there exist n polynomials $\varphi_{kn}(g, d\alpha, x)$ of degree at most $n - 1$ such that*

$$\int_{-\infty}^{\infty} \varphi_{kn}(g, d\alpha, x)\varphi_{ln}(g, d\alpha, x)\, d\alpha(x) = \delta_{kl},$$

$$\int_{-\infty}^{\infty} \varphi_{kn}(g, d\alpha, x)\varphi_{ln}(g, d\alpha, x)g(x)\, d\alpha(x) = \Lambda_{kn}(g, d\alpha)\delta_{kl}$$

and

$$\sum_{k=1}^{n} \varphi_{kn}(g, d\alpha, x)\varphi_{kn}(g, d\alpha, t) = K_n(d\alpha, x, t). \tag{10}$$

PROOF. Let $b_k = (b_k^0, b_k^1, \ldots, b_k^{n-1})$, $k = 1, 2, \ldots, n$, be a collection of orthonormalized eigenvectors of $T_n(g, d\alpha)$, that is $T_n b_k = \Lambda_{kn} b_k$ and $(b_k, b_l) = \delta_{kl}$. Define $\varphi_{kn}(x)$ by $\varphi_{kn}(x) = \sum_{j=0}^{n-1} b_k^j p_j(d\alpha, x)$. Then

$$\int_{-\infty}^{\infty} \varphi_{kn}(x)\varphi_{ln}(x)\, d\alpha(x) = (b_k, b_l) = \delta_{kl} \tag{11}$$

and

$$\int_{-\infty}^{\infty} \varphi_{kn}(x)\varphi_{ln}(x)g(x)\,d\alpha(x) = (T_n b_k, b_l) = \Lambda_{kn}\delta_{kl}.$$

Since the number of the polynomials φ_{kn} is exactly n, the orthogonality relation (11) guarantees that $\sum_{k=1}^{n} \varphi_{kn}(x)\varphi_{kn}(t)$ is a reproduction kernel for the space of polynomials of degree at most $n-1$ supplied with the inner product of $L^2(d\alpha)$. Now (10) follows from the uniqueness of the reproduction kernel.

PROOF OF THEOREM 1. Note that $f(x, t) = G(g(t))$ obviously satisfies (2) of Theorem 2. Thus, since

$$\int_{-1}^{1} G(g(x))K_n(d\alpha, x, x)\,d\alpha(x) = \int_{-1}^{1}\int_{-1}^{1} G(g(t))K_n^2(d\alpha, x, t)\,d\alpha(x)\,d\alpha(t),$$

the theorem will be proved if it can show the validity of

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n} G(\Lambda_{kn}) - \frac{1}{n}\int_{-1}^{1} G(g(x))K_n(d\alpha, x, x)\,d\alpha(x)\right\} = 0. \tag{12}$$

By Lemma 4 we have

$$\frac{1}{n}\sum_{k=1}^{n} G(\Lambda_{kn}) - \frac{1}{n}\int_{-1}^{1} G(g(x))K_n(d\alpha, x, x)\,d\alpha(x)$$
$$= \frac{1}{n}\sum_{k=1}^{n}\int_{-1}^{1}\left[G(\Lambda_{kn}) - G(g(x))\right]\varphi_{kn}^2(x)\,d\alpha(x) \equiv I$$

where $\varphi_{kn}(x) = \varphi_{kn}(g, d\alpha, x)$. Now fix $\varepsilon > 0$ and choose $\delta > 0$ so that $|G(x) - G(y)| < \varepsilon$ for $|x - y| < \delta$. Then we can write

$$I = \frac{1}{n}\sum_{k=1}^{n}\int_{|\Lambda_{kn}-g(x)|<\delta}\left[G(\Lambda_{kn}) - G(g(x))\right]\varphi_{kn}^2(x)\,d\alpha(x)$$
$$+ \frac{1}{n}\sum_{k=1}^{n}\int_{|\Lambda_{kn}-g(x)|>\delta}\left[G(\Lambda_{kn}) - G(g(x))\right]\varphi_{kn}^2(x)\,d\alpha(x) \equiv I^1 + I^2.$$

By the choice of δ

$$|I^1| < \varepsilon\frac{1}{n}\sum_{k=1}^{n}\int_{-1}^{1}\varphi_{kn}^2(x)\,d\alpha(x) = \varepsilon. \tag{13}$$

We also have

$$|I^2| < 2\delta^{-2}\max|G|\frac{1}{n}\sum_{k=1}^{n}\int_{-1}^{1}(g(x) - \Lambda_{kn})^2\varphi_{kn}^2(x)\,d\alpha(x)$$
$$= 2\delta^{-2}\max|G|\frac{1}{n}\sum_{k=1}^{n}\int_{-1}^{1}(g^2(x) - 2g(x)\Lambda_{kn} + \Lambda_{kn}^2)\varphi_{kn}^2(x)\,d\alpha(x).$$

Using Lemma 4 we obtain

$$|I^2| < 2\delta^{-2}\max|G|\frac{1}{n}\left\{\int_{-1}^{1} g^2(x)K_n(d\alpha, x, x)\,d\alpha(x) - \sum_{k=1}^{n}\Lambda_{kn}^2\right\}. \tag{14}$$

Since $K_n(d\alpha, x, x) = \int_{-1}^{1} K_n^2(d\alpha, x, t)\,d\alpha(t)$ and

$$\sum_{k=1}^{n}\Lambda_{kn}^2 = \operatorname{Trace}(T_n^2(g, d\alpha)) = \int_{-1}^{1}\int_{-1}^{1} g(x)g(t)K_n^2(d\alpha, x, t)\,d\alpha(x)\,d\alpha(t),$$

we can rewrite (14) in the form

$$|I^2| \leqslant 2\delta^{-2} \max|G| \frac{1}{n} \int_{-1}^{1} \int_{-1}^{1} [g^2(x) - g(x)g(t)] K_n^2(d\alpha, x, t)\, d\alpha(x)\, d\alpha(t).$$

Note that the function $f(x, t) = g^2(x) - g(x)g(t)$ satisfies condition (2) of Theorem 2 and $f(x, x) = 0$ almost everywhere in $[-1, 1]$. Therefore $\lim_{n\to\infty} I^2 = 0$ holds. Taking (13) into account we get $\limsup_{n\to\infty}|I| \leqslant \varepsilon$. Since ε is arbitrary, I tends to 0 as $n \to \infty$; that is (12) has been proved.

Proof of Theorem 3. It is well known that for every polynomial π

$$\frac{\int_{-1}^{1} \pi^2(t) g(t)\, d\alpha(t)}{\int_{-1}^{1} \pi^2(t)\, d\alpha(t)} \leqslant M. \tag{15}$$

For $-1 < x < 1$ and $n = 1, 2, \ldots,$ let $\pi(t)$ be defined by $\pi(t) = K_n(dv, x, t)$ where dv denotes the Chebyshev measure of $[-1, 1]$. Then by (15) the inequality

$$\frac{\lambda_n(dv, x) \int_{-1}^{1} K_n^2(dv, x, t) g(t)\, d\alpha(t)}{\lambda_n(dv, x) \int_{-1}^{1} K_n^2(dv, x, t)\, d\alpha(t)} \leqslant M \tag{16}$$

holds. It follows from Lemma 6.2.32 of [4, p. 93] that the numerator and denominator in (16) converge to $(1 - x^2)^{1/2} g(x)\alpha'(x)$ and $(1 - x^2)^{1/2}\alpha'(x)$ respectively for almost every $x \in [-1, 1]$ as $n \to \infty$. Since $\alpha'(x) > 0$ almost everywhere in $[-1, 1]$, we obtain $g(x) \leqslant M$. Applying the same reasoning to $T(-g, d\alpha)$ we obtain $g(x) \geqslant m$ almost everywhere in $[-1, 1]$. The last part of the theorem follows from the first one and from Theorem 1.18(b) of [3].

References

1. P. Erdös and P. Turan, *On interpolation*. III, Ann. of Math. (2) **41** (1940), 510–553.
2. G. Freud, *Orthogonale polynome*, Akadémiai Kiadó, Budapest, 1969.
3. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. of California Press, Berkeley and Los Angeles, 1958.
4. P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. No. 213, Amer. Math. Soc., Providence, R. I., 1979.

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