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SOME INEQUALITIES FOR ENTIRE FUNCTIONS

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ABSTRACT. For any entire functions $\varphi(z)$ and $\psi(z)$ on $\mathbf{C}$ with finite norm

$$\left\{\frac{1}{\pi}\iint_{\mathbf{C}}|f(z)|^2\exp(-|z|^2)\,dx\,dy\right\}^{1/2}<\infty,$$

we show that the inequality

$$\frac{2}{\pi}\iint_{\mathbf{C}}|\varphi(z)\psi(z)|^2\exp(-2|z|^2)\,dx\,dy$$
$$\leq\frac{1}{\pi}\iint_{\mathbf{C}}|\varphi(z)|^2\exp(-|z|^2)\,dx\,dy\,\frac{1}{\pi}\iint_{\mathbf{C}}|\psi(z)|^2\exp(-|z|^2)\,dx\,dy$$

holds. This inequality is obtained as a special case of a general result. We also refer to some properties of a tensor product of spaces of entire functions.

1. Introduction. Let $\mathfrak{F}=\mathfrak{F}_1$ denote the Hilbert space (Fischer space) composed of all entire functions $f(z)$ on the complex plane $\mathbf{C}$ with a finite norm

$$\|f\|_1=\left\{\frac{1}{\pi}\iint_{\mathbf{C}}|f(z)|^2\exp(-|z|^2)\,dx\,dy\right\}^{1/2}<\infty\qquad(z=x+iy).\tag{1.1}$$

Cf. [2], [4], [5]. For the case of entire functions on $\mathbf{C}^n=\mathbf{C}\times\mathbf{C}\times\dots\times\mathbf{C}$, our argument in this paper is similar. Hence, for simplicity we consider only the case on $\mathbf{C}$. For any integer n ($n\geq 2$), we introduce the Hilbert space $\mathfrak{F}_n$ composed of all entire functions $F(z)$ on $\mathbf{C}$ with a finite norm

$$\|F\|_n=\left\{\frac{n}{\pi}\iint_{\mathbf{C}}|F(z)|^2\exp(-n|z|^2)\,dx\,dy\right\}^{1/2}<\infty.\tag{1.2}$$

See §3. Then, we shall show the following theorem.

THEOREM 1.1. *Any $F(z)\in\mathfrak{F}_n$ is expressible in a series*

$$F(z)=\sum_{\nu=0}^{\infty}\prod_{j=1}^{n}f_{j,\nu}(z),\qquad f_{j,\nu}(z)\in\mathfrak{F},\tag{1.3}$$

and the equality

$$\frac{n}{\pi}\iint_{\mathbf{C}}|F(z)|^2\exp(-n|z|^2)\,dx\,dy$$
$$=\min\left\{\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\prod_{j=1}^{n}\frac{1}{\pi}\iint_{\mathbf{C}}f_{j,\mu}(z)\overline{f_{j,\nu}(z)}\,\exp(-|z|^2)\,dx\,dy\right\}\tag{1.4}$$

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holds. The minimum is taken here over all analytic functions $\sum_{\nu=0}^{\infty}\prod_{j=1}^{n} f_{j,\nu}(z_j)$ *on* $\mathbf{C}^n$ *satisfying* (1.3).

In particular, for any $f_j(z) \in \mathfrak{F}$, *we obtain*

$$\frac{n}{\pi}\iint_{\mathbf{C}}\left|\prod_{j=1}^{n} f_j(z)\right|^2 \exp(-n|z|^2)\,dx\,dy < \prod_{j=1}^{n}\left\{\frac{1}{\pi}\iint_{\mathbf{C}}|f_j(z)|^2\exp(-|z|^2)\,dx\,dy\right\}. \quad (1.5)$$

Equality holds here if and only if $\prod_{j=1}^{n} f_j(z)$ *is expressible in the form* $C\exp(n\bar{u}z)$ *for some point* $u \in \mathbf{C}$ *and for some constant* C.

Furthermore, we investigate some properties of the tensor (direct) product $\mathfrak{F}_{\otimes}^{n} = \mathfrak{F} \otimes \mathfrak{F} \otimes \cdots \otimes \mathfrak{F}$ as in [7], [8].

2. Preliminary facts. In order to state a background of Theorem 1.1, we consider the tensor product $\mathfrak{F}_{\otimes}^{n}$. Cf. [3, Chapter II]. Further, we consider the Hilbert space $[\mathfrak{F}_{\otimes}^{n}]_r$ which is formed by restricting functions in $\mathfrak{F}_{\otimes}^{n}$ to the diagonal set of $\mathbf{C}^n$ formed by all elements $\{(z, z, \ldots, z)|z \in \mathbf{C}\}$. Here, for any such restriction $F \in [\mathfrak{F}_{\otimes}^{n}]_r$, the norm $\|F\|_{[\mathfrak{F}_{\otimes}^{n}]_r}$ is given by $\min\|H\|_{\mathfrak{F}_{\otimes}^{n}}$ for all H, the restriction of which to the diagonal set is F. See [1, Theorem II, p. 361]. We let $k(z, \bar{u}) = \exp(\bar{u}z)$ denote the reproducing kernel for $\mathfrak{F}$. Cf. [2], [4], [5]. Then, the product $\prod_{j=1}^{n} k(z_j, \bar{u}_j)$ is the reproducing kernel for $\mathfrak{F}_{\otimes}^{n}$ and, on the other hand, $k(z, \bar{u})^n$ is the reproducing kernel for $[\mathfrak{F}_{\otimes}^{n}]_r$ [1, pp. 357–362].

3. Proof of equality of Theorem 1.1. One crucial ingredient in this paper is the observation that $\exp(n\bar{u}z)$ is the reproducing kernel for $\mathfrak{F}_n$. To start with, we show this fact. Let $F(z) \in \mathfrak{F}_n$ be an entire function with the power series $F(z) = \sum_{j=0}^{\infty} A_j z^j$ and we have

$$\frac{n}{\pi}\iint_{\mathbf{C}}|F(z)|^2\exp(-n|z|^2)\,dx\,dy = \sum_{j=0}^{\infty}\frac{j!}{n^j}|A_j|^2. \quad (3.1)$$

In particular, we note that

$$\frac{n}{\pi}\iint_{\mathbf{C}} z^j\bar{z}^k\exp(-n|z|^2)\,dx\,dy = 0 \quad \text{for } j \neq k. \quad (3.2)$$

Thus the simplest orthonormal system for $\mathfrak{F}_n$ is given by

$$\left\{\sqrt{(n^j/j!)}\; z^j\right\}_{j=0}^{\infty}. \quad (3.3)$$

From (3.1), we see easily that $\mathfrak{F}_n$ forms a Hilbert space with the norm (1.2) and this system (3.3) is complete in $\mathfrak{F}_n$. We thus have the reproducing kernel $K(z, \bar{u})$ for $\mathfrak{F}_n$

$$K(z, \bar{u}) = \sum_{j=0}^{\infty}\frac{n^j\bar{u}^j z^j}{j!} = \exp(n\bar{u}z). \quad (3.4)$$

Cf. [2].

We thus have the identity

$$K(z, \bar{u}) = k(z, \bar{u})^n. \quad (3.5)$$

At this point, we apply the theory of reproducing kernels by Aronszajn [1]. Another crucial ingredient is the observation that to every reproducing kernel (or positive definite matrix) $K(p, q)$, there corresponds one and only one class of functions with a uniquely determined quadratic form in it, forming a Hilbert space and admitting $K(p, q)$ as a reproducing kernel (cf. [1, p. 344]). Hence we have the identity

$$\mathfrak{F}_n = [\mathfrak{F}_\otimes^n]_r. \tag{3.6}$$

From this identity, we thus obtain (1.3) and (1.4). Cf. [1, pp. 391–393].

4. Proof of equality statement of Theorem 1.1. First of all, we note that for $\prod_{j=1}^n f_j(z)$, equality holds in (1.4) if and only if

$$\left(\prod_{j=1}^n f_j(z_j), F(z_1, z_2, \ldots, z_n)\right)_{\mathfrak{F}_\otimes^n} = 0$$

$$\text{for all } F \in \mathfrak{F}_\otimes^n \text{ satisfying } F(z, z, \ldots, z) = 0 \text{ on } \mathbf{C}. \tag{4.1}$$

Cf. [6, Equation (3.2)]. Therefore, if for $\prod_{j=1}^n f_j(z)$ equality holds in (1.4), then we have

$$\left(\prod_{j=1}^n f_j(z_j), \prod_{j=1}^n k(z_j, \bar{u}_j) - \prod_{j=1}^n k(z_j, \bar{u}_{\sigma(j)})\right)_{\mathfrak{F}_\otimes^n} = 0, \quad \text{for all } u_j \in \mathbf{C}, \tag{4.2}$$

where σ is any permutation

$$\sigma = \begin{pmatrix} 1, & 2, & \ldots, & n \\ \sigma(1), & \sigma(2), & \ldots, & \sigma(n) \end{pmatrix}.$$

We thus see that any $f_j(z)$ and $f_k(z)$ $(j \neq k)$ are linearly dependent. Hence, we can set $f_j(z) = f(z)$ for all j. From (4.1), on the other hand, we have

$$\left(\prod_{j=1}^n f_j(z_j), \left[k(z_1, \bar{u})k(z_2, \bar{v}) - k(z_1, \bar{u})k(z_1, \bar{v}) \times 1\right] \prod_{j=3}^n k(z_j, \bar{u}_j)\right)_{\mathfrak{F}_\otimes^n} = 0 \tag{4.3}$$

and so

$$\left\{ f(u)f(v) - \frac{1}{\pi} \iint_{\mathbf{C}} f(z_2)\exp(-|z_2|^2)\, dx_2\, dy_2 \right.$$
$$\left. \times \frac{1}{\pi} \iint_{\mathbf{C}} f(z_1)\, \overline{k(z_1, \bar{u})k(z_1, \bar{v})}\exp(-|z_1|^2)\, dx_1\, dy_1 \right\} \prod_{j=3}^n f(u_j) = 0,$$

$$\text{for all } u, v, u_j \in \mathbf{C}, z_j = x_j + iy_j. \tag{4.4}$$

Hence, for $f \not\equiv 0$ we obtain

$$f(u)f(v) = \frac{1}{\pi} \iint_{\mathbf{C}} f(z)\exp(-|z|^2)\, dx\, dy \frac{1}{\pi} \iint_{\mathbf{C}} f(z)\, \overline{k(z, \bar{u})k(z, \bar{v})}\exp(-|z|^2)\, dx\, dy$$

$$\text{for all } u, v \in \mathbf{C}. \tag{4.5}$$

We note that $k(z, \bar{u})k(z, \bar{v}) = k(z, \bar{u} + \bar{v})$ and so we have

$$f(u)f(v) = \left\{ \frac{1}{\pi} \iint_{\mathbf{C}} f(z)\exp(-|z|^2)\, dx\, dy \right\} f(u + v) \quad \text{for all } u, v \in \mathbf{C}. \tag{4.6}$$

From this functional equation, we have the expression $f(z) = C\exp(\bar{u}z) = Ck(z, \bar{u})$ for some point $u \in \mathbf{C}$ and for some constant C.

On the other hand, since the functions $C\prod_{j=1}^{n} k(z_j, \bar{u})$ on $\mathbf{C}^n$ satisfy (4.1), we have the desired result.

5. Integral transform by $\prod_{j=1}^{n} k(z, \bar{z}_j)$. In this section, we discuss an analogue of [7], [8] for $\mathfrak{F}_{\otimes}^{n}$. However, in this case the circumstances are quite different. We let $[\mathfrak{F}_{\otimes}^{n}]_{D(0)}$ denote the subspace in $\mathfrak{F}_{\otimes}^{n}$ composed of all functions $F(z_1, z_2, \ldots, z_n)$ in $\mathfrak{F}_{\otimes}^{n}$ such that $F(z, z, \ldots, z) = 0$ on $\mathbf{C}$. Let $([\mathfrak{F}_{\otimes}^{n}]_{D(0)})^{\perp}$ be the orthocomplement of $[\mathfrak{F}_{\otimes}^{n}]_{D(0)}$ in $\mathfrak{F}_{\otimes}^{n}$.

First, from Theorem 1.1 and the sentence containing (4.1), we obtain

THEOREM 5.1. *For any $F \in \mathfrak{F}_{\otimes}^{n}$, we have the inequality*

$$\frac{1}{\pi^n} \iint_{\mathbf{C}} \cdots \iint_{\mathbf{C}} |F(z_1, z_2, \ldots, z_n)|^2 \times \exp\{-(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)\}\, dx_1\, dy_1 \cdots dx_n\, dy_n \geq \frac{n}{\pi} \iint_{\mathbf{C}} |F(z, z, \ldots, z)|^2 \exp(-n|z|^2)\, dx\, dy.$$

Equality holds here if and only if $F \in ([\mathfrak{F}_{\otimes}^{n}]_{D(0)})^{\perp}$.

Next, we shall give an integral representation of an important subspace $([\mathfrak{F}_{\otimes}^{n}]_{D(0)})^{\perp}$ in $\mathfrak{F}_{\otimes}^{n}$. Cf. [7, Theorem 4.2] and [8, Theorem 4.1].

THEOREM 5.2. *Any $F \in ([\mathfrak{F}_{\otimes}^{n}]_{D(0)})^{\perp}$ is expressible in the following integral*

$$F(z_1, z_2, \ldots, z_n) = \frac{n}{\pi} \iint_{\mathbf{C}} H(z) \overline{\left\{ \prod_{j=1}^{n} k(z, \bar{z}_j) \right\}} \exp(-n|z|^2)\, dx\, dy \tag{5.1}$$

for a uniquely determined $H(z) \in \mathfrak{F}_n$.

Moreover, $H(z)$ is given by the restriction $F(z, z, \ldots, z)$.

PROOF. From the relation (3.5), for any $H(z) \in \mathfrak{F}_n$ we have $H(z) = F(z, z, \ldots, z)$. Moreover, from the identity (3.6), for any $F \in \mathfrak{F}_{\otimes}^{n}$, we have that $F(z, z, \ldots, z) \in \mathfrak{F}_n$. Since the uniqueness of $H(z)$ is apparent, it is sufficient to prove that for any $H(z) \in \mathfrak{F}_n$

$$\tilde{F}(z_1, z_2, \ldots, z_n) = \frac{n}{\pi} \iint_{\mathbf{C}} H(z) \overline{\left\{ \prod_{j=1}^{n} k(z, \bar{z}_j) \right\}} \exp(-n|z|^2)\, dx\, dy \in ([\mathfrak{F}_{\otimes}^{n}]_{D(0)})^{\perp}. \tag{5.2}$$

We expand $\tilde{F}$ and any $\tilde{\tilde{F}} \in [\mathfrak{F}_{\otimes}^{n}]_{D(0)}$ as follows:

$$\tilde{F}(z_1, z_2, \ldots, z_n) = \sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} \frac{n}{\pi} \iint_{\mathbf{C}} H(z) \overline{\Phi_{j_1}(z)\Phi_{j_2}(z) \cdots \Phi_{j_n}(z)} \times \exp(-n|z|^2)\, dx\, dy\, \Phi_{j_1}(z_1)\Phi_{j_2}(z_2) \cdots \Phi_{j_n}(z_n) \tag{5.3}$$

and

$$\tilde{\tilde{F}}(z_1, z_2, \ldots, z_n) = \sum_{j_1 j_2, \ldots, j_n = 0}^{\infty} A_{j_1 j_2, \ldots, j_n} \Phi_{j_1}(z_1)\Phi_{j_2}(z_2) \cdots \Phi_{j_n}(z_n). \quad (5.4)$$

Here $\{\Phi_j(z)\}_{j=0}^{\infty}$ denotes a complete orthonormal system for $\mathfrak{F}$. From Theorem 1.1, we see that the series

$$\tilde{\tilde{F}}(z, z, \ldots, z) = \sum_{j_1 j_2, \ldots, j_n = 0}^{\infty} A_{j_1 j_2, \ldots, j_n} \Phi_{j_1}(z)\Phi_{j_2}(z) \cdots \Phi_{j_n}(z) \quad \text{on } \mathbf{C}$$

also converges in the sense of the $\mathfrak{F}_n$-norm. We thus obtain

$$\begin{aligned}(\tilde{\tilde{F}}, \tilde{F})_{\mathfrak{F}_\otimes^n} &= \sum_{j_1 j_2, \ldots, j_n = 0}^{\infty} A_{j_1 j_2, \ldots, j_n} \\ &\quad \times \overline{\left\{ \frac{n}{\pi} \iint_{\mathbf{C}} H(z)\, \overline{\Phi_{j_1}(z)\Phi_{j_2}(z) \cdots \Phi_{j_n}(z)}\, \exp(-n|z|^2)\, dx\, dy \right\}} \\ &= \frac{n}{\pi} \iint_{\mathbf{C}} \overline{H(z)} \left\{ \sum_{j_1 j_2, \ldots, j_n = 0}^{\infty} A_{j_1 j_2, \ldots, j_n} \Phi_{j_1}(z)\Phi_{j_2}(z) \cdots \Phi_{j_n}(z) \right\} \\ &\qquad \times \exp(-n|z|^2)\, dx\, dy \\ &= \frac{n}{\pi} \iint_{\mathbf{C}} \overline{H(z)}\, \tilde{\tilde{F}}(z, z, \ldots, z) \exp(-n|z|^2)\, dx\, dy = 0, \end{aligned} \quad (5.5)$$

which implies the desired assertion.

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