SOME INEQUALITIES FOR ENTIRE FUNCTIONS

SABUROU SAITOH

ABSTRACT. For any entire functions \( \varphi(z) \) and \( \psi(z) \) on \( \mathbb{C} \) with finite norm
\[
\left\{ \frac{1}{\pi} \int \int |f(z)|^2 \exp(-|z|^2) \, dx \, dy \right\}^{1/2} < \infty,
\]
we show that the inequality
\[
\frac{2}{\pi} \int \int |\varphi(z)\psi(z)|^2 \exp(-2|z|^2) \, dx \, dy < \frac{1}{\pi} \int \int |\varphi(z)|^2 \exp(-|z|^2) \, dx \, dy \frac{1}{\pi} \int \int |\psi(z)|^2 \exp(-|z|^2) \, dx \, dy
\]
holds. This inequality is obtained as a special case of a general result. We also refer to some properties of a tensor product of spaces of entire functions.

1. Introduction. Let \( \mathcal{F} = \mathcal{F}_1 \) denote the Hilbert space (Fischer space) composed of all entire functions \( f(z) \) on the complex plane \( \mathbb{C} \) with a finite norm
\[
\|f\|_1 = \left\{ \frac{1}{\pi} \int \int |f(z)|^2 \exp(-|z|^2) \, dx \, dy \right\}^{1/2} < \infty \quad (z = x + iy). \tag{1.1}
\]
Cf. [2], [4], [5]. For the case of entire functions on \( \mathbb{C}^n = \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C} \), our argument in this paper is similar. Hence, for simplicity we consider only the case on \( \mathbb{C} \). For any integer \( n \) (\( n > 2 \)), we introduce the Hilbert space \( \mathcal{F}_n \) composed of all entire functions \( F(z) \) on \( \mathbb{C} \) with a finite norm
\[
\|F\|_n = \left\{ \frac{n}{\pi} \int \int |F(z)|^2 \exp(-n|z|^2) \, dx \, dy \right\}^{1/2} < \infty. \tag{1.2}
\]
See §3. Then, we shall show the following theorem.

**Theorem 1.1.** Any \( F(z) \in \mathcal{F}_n \) is expressible in a series
\[
F(z) = \sum_{\nu=0}^{\infty} \prod_{j=1}^{n} f_{\nu,j}(z), \quad f_{\nu,j}(z) \in \mathcal{F}, \tag{1.3}
\]
and the equality
\[
\frac{n}{\pi} \int \int |F(z)|^2 \exp(-n|z|^2) \, dx \, dy = \min \left\{ \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \prod_{j=1}^{n} \frac{1}{\pi} \int \int f_{\nu,j}(z) f_{\mu,j}(z) \exp(-|z|^2) \, dx \, dy \right\} \tag{1.4}
\]

Received by the editors May 30, 1979 and, in revised form, October 26, 1979.
1980 Mathematics Subject Classification. Primary 30C40; Secondary 30A10, 30D20.

Key words and phrases. Space of entire functions, Fischer space, inequality for entire functions, tensor product of Hilbert spaces, general theory of reproducing kernels.

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holds. The minimum is taken here over all analytic functions \( \sum_{\nu=0}^{\infty} \prod_{j=1}^{n} f_{\nu}(z_j) \) on \( \mathbb{C}^n \) satisfying (1.3).

In particular, for any \( f_j(z) \in \mathcal{F} \), we obtain

\[
\frac{n}{\pi} \int_{\mathbb{C}} \left| \prod_{j=1}^{n} f_j(z) \right|^2 \exp(-n|z|^2) \, dx \, dy < \prod_{j=1}^{n} \left\{ \frac{1}{\pi} \int_{\mathbb{C}} |f_j(z)|^2 \exp(-|z|^2) \, dx \, dy \right\}.
\]

(1.5)

Equality holds here if and only if \( \prod_{j=1}^{n} f_j(z) \) is expressible in the form \( C \exp(nu \bar{z}) \) for some point \( u \in \mathbb{C} \) and for some constant \( C \).

Furthermore, we investigate some properties of the tensor (direct) product \( \mathcal{F}_n^\otimes = \mathcal{F} \otimes \mathcal{F} \otimes \cdots \otimes \mathcal{F} \) as in [7], [8].

2. Preliminary facts. In order to state a background of Theorem 1.1, we consider the tensor product \( \mathcal{F}_n^\otimes \). Cf. [3, Chapter II]. Further, we consider the Hilbert space \([\mathcal{F}_n^\otimes]\), which is formed by restricting functions in \( \mathcal{F}_n^\otimes \) to the diagonal set of \( \mathbb{C}^n \) formed by all elements \( \{(z, z, \ldots, z) | z \in \mathbb{C}\} \). Here, for any such restriction \( F \in [\mathcal{F}_n^\otimes], \) the norm \( \|F\|_{[\mathcal{F}_n^\otimes]} \) is given by \( \min \|H\|_{[\mathcal{F}_n]} \) for all \( H \), the restriction of which to the diagonal set is \( F \). See [1, Theorem II, p. 361]. We let \( k(z, \bar{u}) = \exp(\bar{u}z) \) denote the reproducing kernel for \( \mathcal{F} \). Cf. [2], [4], [5]. Then, the product \( \prod_{j=1}^{n} k(z_j, \bar{u}_j) \) is the reproducing kernel for \( \mathcal{F}_n^\otimes \) and, on the other hand, \( k(z, \bar{u})^n \) is the reproducing kernel for \([\mathcal{F}_n^\otimes]\), [1, pp. 357–362].

3. Proof of equality of Theorem 1.1. One crucial ingredient in this paper is the observation that \( \exp(nu \bar{z}) \) is the reproducing kernel for \( \mathcal{F}_n^\otimes \). To start with, we show this fact. Let \( F(z) \in \mathcal{F}_n^\otimes \) be an entire function with the power series \( F(z) = \sum_{\nu=0}^{\infty} A_{\nu} z^\nu \) and we have

\[
\frac{n}{\pi} \int_{\mathbb{C}} |F(z)|^2 \exp(-n|z|^2) \, dx \, dy = \sum_{\nu=0}^{\infty} \frac{1}{n!} |A_{\nu}|^2.
\]

(3.1)

In particular, we note that

\[
\frac{n}{\pi} \int_{\mathbb{C}} z^j \bar{z}^k \exp(-n|z|^2) \, dx \, dy = 0 \quad \text{for } j \neq k.
\]

(3.2)

Thus the simplest orthonormal system for \( \mathcal{F}_n^\otimes \) is given by

\[
\left\{ \sqrt{n!/j!} \, z^j \right\}_{j=0}^{\infty}.
\]

(3.3)

From (3.1), we see easily that \( \mathcal{F}_n^\otimes \) forms a Hilbert space with the norm (1.2) and this system (3.3) is complete in \( \mathcal{F}_n^\otimes \). We thus have the reproducing kernel \( K(z, \bar{u}) \) for \( \mathcal{F}_n^\otimes \)

\[
K(z, \bar{u}) = \sum_{j=0}^{\infty} \frac{n^j \bar{u}^j z^j}{j!} = \exp(nu \bar{z}).
\]

(3.4)

Cf. [2].

We thus have the identity

\[
K(z, \bar{u}) = k(z, \bar{u})^n.
\]

(3.5)
At this point, we apply the theory of reproducing kernels by Aronszajn [1]. Another crucial ingredient is the observation that to every reproducing kernel (or positive definite matrix) $K(p, q)$, there corresponds one and only one class of functions with a uniquely determined quadratic form in it, forming a Hilbert space and admitting $K(p, q)$ as a reproducing kernel (cf. [1, p. 344]). Hence we have the identity
$$\mathcal{F}_n = \left[ \mathcal{F}^n_{\infty} \right].$$
From this identity, we thus obtain (1.3) and (1.4). Cf. [1, pp. 391–393].

4. Proof of equality statement of Theorem 1.1. First of all, we note that for $\prod_{j=1}^n f_j(z)$, equality holds in (1.4) if and only if
$$\left( \prod_{j=1}^n f_j(z_j), \prod_{j=1}^n k(z_j, \bar{u}_j) - \prod_{j=1}^n k(z_j, \bar{u}_{a(j)}) \right)_{\mathcal{H}_n} = 0, \quad \text{for all } u_j \in \mathbb{C}, \quad (4.2)$$
where $\sigma$ is any permutation
$$\sigma = \left( 1, 2, \ldots, n \right).$$
We thus see that any $f_j(z)$ and $f_k(z)$ ($j \neq k$) are linearly dependent. Hence, we can set $f_j(z) = f(z)$ for all $j$. From (4.1), on the other hand, we have
$$\left( \prod_{j=1}^n f_j(z_j), \left[ k(z_1, \bar{u})k(z_2, \bar{v}) - k(z_1, \bar{u})k(z_1, \bar{v}) \times 1 \right] \prod_{j=3}^n k(z_j, \bar{u}_j) \right)_{\mathcal{H}_n} = 0 \quad (4.3)$$
and so
$$\left\{ f(u)f(v) - \frac{1}{\pi} \int_{\mathcal{C}} f(z_2) \exp(-|z_2|^2) \, dx_2 \, dy_2 \right. \times \left. \frac{1}{\pi} \int_{\mathcal{C}} f(z_1) \, k(z_1, \bar{u})k(z_1, \bar{v}) \exp(-|z_1|^2) \, dx_1 \, dy_1 \right\} \prod_{j=3}^n f(u_j) = 0,$$
for all $u, v, u_j \in \mathbb{C}, z_j = x_j + iy_j$. \quad (4.4)
Hence, for $f \equiv 0$ we obtain
$$f(u)f(v) = \frac{1}{\pi} \int_{\mathcal{C}} f(z) \exp(-|z|^2) \, dx \, dy \frac{1}{\pi} \int_{\mathcal{C}} f(z) \, k(z, \bar{u})k(z, \bar{v}) \exp(-|z|^2) \, dx \, dy$$
for all $u, v \in \mathbb{C}$. \quad (4.5)
We note that $k(z, \bar{u})k(z, \bar{v}) = k(z, \bar{u} + \bar{v})$ and so we have
$$f(u)f(v) = \left\{ \frac{1}{\pi} \int_{\mathcal{C}} f(z) \exp(-|z|^2) \, dx \, dy \right\} f(u + v) \quad \text{for all } u, v \in \mathbb{C}. \quad (4.6)$$
From this functional equation, we have the expression \( f(z) = C \exp(\bar{u}z) = Ck(z, \bar{u}) \) for some point \( u \in \mathbb{C} \) and for some constant \( C \).

On the other hand, since the functions \( C \prod_{j=1}^{n-1} k(z_j, \bar{u}) \) on \( \mathbb{C}^n \) satisfy (4.1), we have the desired result.

5. Integral transform by \( \prod_{j=1}^{n-1} k(z, \bar{z}_j) \). In this section, we discuss an analogue of [7], [8] for \( \mathcal{T}_n^\circ \). However, in this case the circumstances are quite different. We let \( (\mathcal{T}_n^\circ)_{D(0)} \) denote the subspace in \( \mathcal{T}_n^\circ \) composed of all functions \( F(z_1, z_2, \ldots, z_n) \) in \( \mathcal{T}_n^\circ \) such that \( F(z, z, \ldots, z) = 0 \) on \( \mathbb{C} \). Let \( (\mathcal{T}_n^\circ)_{D(0)}^\perp \) be the orthocomplement of \( (\mathcal{T}_n^\circ)_{D(0)} \) in \( \mathcal{T}_n^\circ \).

First, from Theorem 1.1 and the sentence containing (4.1), we obtain

**THEOREM 5.1.** For any \( F \in \mathcal{T}_n^\circ \), we have the inequality

\[
\frac{1}{\pi^n} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \left| F(z_1, z_2, \ldots, z_n) \right|^2 \exp\left\{ -\left( |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \right) \right\} \, dx_1 \, dy_1 \cdots dx_n \, dy_n \geq \frac{n}{\pi} \int_{\mathbb{C}} \left| F(z, z, \ldots, z) \right|^2 \exp(-n|z|^2) \, dx \, dy.
\]

Equality holds here if and only if \( F \in (\mathcal{T}_n^\circ)_{D(0)}^\perp \).

Next, we shall give an integral representation of an important subspace \( (\mathcal{T}_n^\circ)_{D(0)}^\perp \) in \( \mathcal{T}_n^\circ \). Cf. [7, Theorem 4.2] and [8, Theorem 4.1].

**THEOREM 5.2.** Any \( F \in (\mathcal{T}_n^\circ)_{D(0)}^\perp \) is expressible in the following integral

\[
F(z_1, z_2, \ldots, z_n) = \frac{n}{\pi} \int_{\mathbb{C}} H(z) \left\{ \prod_{j=1}^{n} k(z, \bar{z}_j) \right\} \exp(-n|z|^2) \, dx \, dy
\]

for a uniquely determined \( H(z) \in \mathcal{T}_n^\circ \).

Moreover, \( H(z) \) is given by the restriction \( F(z, z, \ldots, z) \).

**PROOF.** From the relation (3.5), for any \( H(z) \in \mathcal{T}_n \) we have \( H(z) = F(z, z, \ldots, z) \). Moreover, from the identity (3.6), for any \( F \in \mathcal{T}_n^\circ \), we have that \( F(z, z, \ldots, z) \in \mathcal{T}_n \). Since the uniqueness of \( H(z) \) is apparent, it is sufficient to prove that for any \( H(z) \in \mathcal{T}_n \)

\[
\tilde{F}(z_1, z_2, \ldots, z_n) = \frac{n}{\pi} \int_{\mathbb{C}} H(z) \left\{ \prod_{j=1}^{n} k(z, \bar{z}_j) \right\} \exp(-n|z|^2) \, dx \, dy \in (\mathcal{T}_n^\circ)_{D(0)}^\perp.
\]

We expand \( \tilde{F} \) and any \( \tilde{F} \in (\mathcal{T}_n^\circ)_{D(0)} \) as follows:

\[
\tilde{F}(z_1, z_2, \ldots, z_n) = \sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} \frac{n}{\pi} \int_{\mathbb{C}} H(z) \Phi_{j_1}(z) \Phi_{j_2}(z) \cdots \Phi_{j_n}(z) \exp(-n|z|^2) \, dx \, dy \Phi_{j_1}(z_1) \Phi_{j_2}(z_2) \cdots \Phi_{j_n}(z_n).
\]
and
\[ \tilde{F}(z_1, z_2, \ldots, z_n) = \sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} A_{j_1, j_2, \ldots, j_n} \Phi_{j_1}(z_1) \Phi_{j_2}(z_2) \cdots \Phi_{j_n}(z_n). \quad (5.4) \]

Here \( \{\Phi_j(z)\}_{j=0}^{\infty} \) denotes a complete orthonormal system for \( \mathcal{F} \). From Theorem 1.1, we see that the series
\[ \tilde{F}(z, z, \ldots, z) = \sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} A_{j_1, j_2, \ldots, j_n} \Phi_{j_1}(z) \Phi_{j_2}(z) \cdots \Phi_{j_n}(z) \quad \text{on } \mathbb{C} \]
also converges in the sense of the \( \mathcal{F}_n \)-norm. We thus obtain
\[
(\tilde{F}, \tilde{F})_n^* = \sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} A_{j_1, j_2, \ldots, j_n} \times \left\{ \frac{n}{\pi} \int \int_{\mathbb{C}} H(z) \Phi_{j_1}(z) \Phi_{j_2}(z) \cdots \Phi_{j_n}(z) \exp(-n|z|^2) \, dx \, dy \right\}
\]
\[
= \frac{n}{\pi} \int \int_{\mathbb{C}} H(z) \left\{ \sum_{j_1, j_2, \ldots, j_n = 0}^{\infty} A_{j_1, j_2, \ldots, j_n} \Phi_{j_1}(z) \Phi_{j_2}(z) \cdots \Phi_{j_n}(z) \right\} \times \exp(-n|z|^2) \, dx \, dy
\]
\[
= \frac{n}{\pi} \int \int_{\mathbb{C}} H(z) \tilde{F}(z, z, \ldots, z) \exp(-n|z|^2) \, dx \, dy = 0, \quad (5.5)
\]
which implies the desired assertion.

ACKNOWLEDGMENT. The author gratefully acknowledges the help of the referee in improving the representation of this material.

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Department of Mathematics, Faculty of Engineering, Gunma University 1-5-1, Tenjin-Cho, Kiryu 376, Japan