A NOTE ON HADAMARD PRODUCTS OF UNIVALENT FUNCTIONS

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Abstract. An example is constructed to show that a modified Hadamard product of two normalized univalent functions with real coefficients may not be univalent.

Let $S$ denote the class of all functions $f(z) = z + c_2z^2 + \cdots$, analytic and univalent in the unit disk. Given two functions in $S$, $f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define their modified Hadamard product by

$$(f_1 \ast f_2)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n.$$ 

Let $S_R$ be the set of functions in $S$ with real coefficients. In [1] Krzyz questions whether this modified Hadamard product of two functions in $S_R$ is in $S_R$. The following argument leads to a counterexample. It depends on a weak version of a theorem of Jenkins (see [2, p. 120, Corollary 4.8 and Example 4.5]).

Theorem. Let $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $S_R$, and $0 < \lambda < 2$. If

$$\alpha_2 = \lambda(1 + \log(2/\lambda)) \equiv x(\lambda),$$

then

$$\alpha_3 < 1 + \frac{1}{4} \lambda^2 + \lambda^2 \left( \frac{1}{2} + \log(2/\lambda) \right)^2 \equiv Y(\lambda) = y(x).$$

For every choice of $x$, there exists $g_x(z)$ in $S_R$ for which equality holds in (2).

In fact, given $0 < x_1 < 2$, then $h = g_{x_1} \ast g_{x_2}$ is not in $S_R$ for $x_2$ sufficiently close to 2.

Note that $h(z) = z + x_1 x_2 z^2/2 + y(x_1) y(x_2) z^3/3 + \cdots$, and if it were in $S_R$, then

$$y(x_1) y(x_2)/3 < y(x_1 x_2/2).$$

Fix $x_1$, and put $x_2 = 2 - r$, for $0 < r < 2$, then

$$\frac{y(x_1)}{3} (y(2) - ry'(2) + o(r)) < y(x_1) - \frac{r x_1}{2} y'(x_1) + o(r).$$

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$y(2) = 3$ and from (1) and (2), $y'(x) = 2\lambda \log(2/\lambda)$. We remain with

$$r \frac{x_1}{2} y'(x_1) + o(r) < 0$$

which leads to a contradiction for small values of $r$.

REFERENCES


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