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A NOTE ON HADAMARD PRODUCTS OF UNIVALENT FUNCTIONS

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ABSTRACT. An example is constructed to show that a modified Hadamard product of two normalized univalent functions with real coefficients may not be univalent.

Let S denote the class of all functions $f(z) = z + c_2z^2 + \cdots$, analytic and univalent in the unit disk. Given two functions in S, $f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define their modified Hadamard product by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n.$$

Let S_R be the set of functions in S with real coefficients. In [1] Krzyz questions whether this modified Hadamard product of two functions in S_R is in S_R. The following argument leads to a counterexample. It depends on a weak version of a theorem of Jenkins (see [2, p. 120, Corollary 4.8 and Example 4.5]).

THEOREM. *Let* $g(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n$ *be in* S_R, *and* $0 < \lambda < 2$. *If*

$$\alpha_2 = \lambda(1 + \log(2/\lambda)) \equiv x(\lambda), \tag{1}$$

then

$$\alpha_3 \leq 1 + \tfrac{1}{4}\lambda^2 + \lambda^2\left(\tfrac{1}{2} + \log(2/\lambda)\right)^2 \equiv Y(\lambda) = y(x). \tag{2}$$

For every choice of x, there exists $g_x(z)$ in S_R for which equality holds in (2).

In fact, given $0 < x_1 < 2$, then $h = g_{x_1} * g_{x_2}$ is not in S_R for x_2 sufficiently close to 2.

Note that $h(z) = z + x_1x_2z^2/2 + y(x_1)y(x_2)z^3/3 + \cdots$, and if it were in S_R, then

$$y(x_1)y(x_2)/3 \leq y(x_1x_2/2).$$

Fix x_1, and put $x_2 = 2 - r$, for $0 < r < 2$, then

$$\frac{y(x_1)}{3}(y(2) - ry'(2) + o(r)) \leq y(x_1) - \frac{rx_1}{2}y'(x_1) + o(r).$$

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$y(2) = 3$ and from (1) and (2), $y'(x) = 2\lambda \log(2/\lambda)$. We remain with

$$r\frac{x_1}{2}y'(x_1) + o(r) \leqslant 0$$

which leads to a contradiction for small values of r.

References

1. J. M. Anderson, K. F. Barth and D. A. Brannan, *Research problems in complex analysis*, Bull. London Math. Soc. 9 (1977), 129–162.

2. Ch. Pommerenke, *Univalent functions*, Vanderhock and Ruprecht, Göttingen, 1975.

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