A GLOBAL EXISTENCE THEOREM
FOR SMOLUCHOWSKI'S COAGULATION EQUATIONS

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Abstract. A countable system of nonlinear ordinary differential equations is shown to admit solutions valid over all positive time.

Consider the following countable system of equations on $[0, \infty)$, $k = 1, 2, \ldots$ [1], [2], [3]:

\[
\frac{dx_k}{dt} - c_k = \frac{1}{2} \sum_{i+j=k} b_{ij}x_i x_j - x_k \sum_j b_{kj}x_j, \tag{1a}
\]

\[
x_k(0) = a_k > 0, \tag{1b}
\]

where the $b_{ij}$ and $c_k$ are continuous functions of $t$ such that $b_{ij} = b_{ji} > 0$ and $c_k > 0$. The $b_{ij}$ arising in applications typically increase without bound as $i$ and $j$ increase, so that (1) is beyond the reach of standard results on differential equations in Banach spaces. The case $b_{ij} < ij$ has been studied by McLeod [4], [5], who established the existence and uniqueness of a nonnegative ($x_k > 0$) initial solution which does not, in general, extend to $t > 1$. The $b_{ij}$ arising in most applications satisfy the more restrictive condition $b_{ij} < i + j$, for which case the present note demonstrates the existence of global, well-behaved solutions.

**Theorem 1.** Let $b_{ij} < i^a + j^a$ for some $a \in [0, 1]$. Let $\Sigma_k k^p a_k$ be finite, and $\Sigma_k k^p c_k$ bounded on bounded intervals, for some integer $p > a$. Then (1) admits at least one nonnegative solution on $[0, \infty)$. For any nonnegative solution $x$, $\Sigma_k k^p x_k$ is bounded on bounded intervals.

The proof of Theorem 1 begins with a technical definition abstracting conditions necessary to carry through McLeod's local arguments [4] on a global basis. It will be convenient to denote sequences and kernels by the unsubscripted symbols used for their elements (e.g., $(a_k) = a$, $(b_{ij}) = b$), and to write $b' < b$ to indicate that $b_{ij}' < b_{ij}$ for all $i$ and $j$. With these conventions, let $S_x(b, c)$ be the (possibly empty) set of nonnegative solutions to (1) on $[0, \infty)$.

**Definition.** Problem (1) has the bounded $\beta$-moment property for some $\beta \in [0, \infty)$ if there exists a continuous function $m_\beta$: $[0, \infty) \to [0, \infty)$ such that $\Sigma_k k^\beta x_k < m_\beta$ for all $x \in \bigcup_{b' < b} S_x(b', c)$.

**Lemma 2.** Let $b_{ij} < Bi^{a+}j^{a+}$ for some $a \in [0, \infty)$. If (1) has the bounded $\beta$-moment property for some $\beta > a$, then it admits at least one nonnegative solution on $[0, \infty)$.

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273
Proof. The proof of this lemma closely follows the approach taken by McLeod in the proof of his Theorem 2 [4].

For each positive integer \( N \), define a truncated kernel \( b^{(N)} \) as follows:

\[
b^{(N)}_{ij} = \begin{cases} b_{ij} & \text{for } i + j < N, \\ 0 & \text{for } i + j > N. \end{cases}
\]

Straightforward finite-dimensional arguments show the solution set \( S(x^{(N)}, c) \) to consist of a single element, \( x^{(N)} \). Assuming (1) to have the bounded \( \beta \)-moment property, there exists a continuous function \( m_\beta: [0, \infty) \to [0, \infty) \) such that \( m_\beta > \sum_k k^\beta x_k^{(N)} \) for each \( N \). Since the \( x^{(N)} \) are nonnegative, \( x_k^{(N)} \) and \( dx_k^{(N)}/dt \) are then bounded on bounded intervals, uniformly for all \( N \):

\[
x_k^{(N)} < \sum_j j^\beta x_j^{(N)} < m_\beta,
\]

\[
\left| \frac{d}{dt} x_k^{(N)} \right| = \left| \frac{1}{2} \sum_{i+j=k} b^{(N)}_{ij} x_i^{(N)} x_j^{(N)} - x_k^{(N)} \sum_j b_k^{(N)} x_j^{(N)} + c_k \right|
\leq \frac{B}{2} \sum_i \sum_j i^\beta j^\beta x_i^{(N)} x_j^{(N)} + Bk^\beta x_k^{(N)} \sum_j j^\beta x_j^{(N)} + c_k < 2Bm_\beta^2 + c_k.
\]

Ascoli's lemma can therefore be applied, repeatedly, to obtain an increasing sequence \( J \) of integers such that \( \{x_k^{(N)}: N \in J\} \) is uniformly convergent on bounded intervals for each \( k \). In what follows, \( N \) will be restricted to values in \( J \).

Let \( x_k = \lim_{N \to \infty} x_k^{(N)} \) for each \( k \). To verify equation (1a) in the limit, it suffices to show that \( \sum_j b_{kj}^{(N)} x_j^{(N)} \) converges to \( \sum_j b_{kj} x_j \) uniformly on bounded intervals. For each \( k \),

\[
\left| \sum_j b_{kj} x_j - \sum_j b_{kj}^{(N)} x_j^{(N)} \right|
\leq \sum_{j < N_1} |b_{kj} x_j - b_{kj}^{(N)} x_j^{(N)}| + \sum_{N_1 < j} (b_{kj} x_j + b_{kj}^{(N)} x_j^{(N)}).
\]

The second term on the right-hand side can be made arbitrarily small by choosing \( N_1 \) large:

\[
\sum_{N_1 < j} (b_{kj} x_j + b_{kj}^{(N)} x_j^{(N)}) < Bk^\alpha \sum_{N_1 < j} j^\alpha (x_j + x_j^{(N)})
\leq Bk^\alpha \sum_{N_1 < j} N_1^{\alpha-\beta} (x_j + x_j^{(N)}) < 2Bk^\alpha m_\beta N_1^{\alpha-\beta}.
\]

The first term on the right-hand side of (2) can be made arbitrarily small— for fixed \( N_1 \)— by choosing \( N \) large.

Proposition 3. Let \( x \) be a nonnegative solution to (1). Then

\[
\frac{d}{dt} \sum_{k < N} k^p x_k - \sum_{k < N} k^p c_k < \frac{1}{2} \sum_{1 < q < p-1} \binom{p}{q} \sum_{i < N} \sum_{j < N} i^q j^{p-q} b_{ij} x_i x_j
\]

for all positive integers \( p \) and \( N \).
Proof. From equation (1a),
\[
\frac{d}{dt} \sum_{k \leq N} k^p x_k - \sum_{k \leq N} k^p c_k = \sum_{k \leq N} k^p \left( \frac{dx_k}{dt} - c_k \right)
\]
\[
= \frac{1}{2} \sum_{k \leq N} k^p \sum_{i+j=k} b_{ij} x_i x_j - \sum_{k \leq N} k^p x_k \sum_{j} b_{kj} x_j
\]
\[
= \frac{1}{2} \sum_{k \leq N} \sum_{i+j=k} (i+j)^p b_{ij} x_i x_j - \sum_{k \leq N} k^p b_{kj} x_k x_j
\]
\[
= \frac{1}{2} \sum_{k \leq N} \sum_{i+j=k} \sum_{1 \leq q \leq p-1} \binom{p}{q} i^q j^{p-q} b_{ij} x_i x_j
\]
\[
+ \sum_{k \leq N} \sum_{i+j=k} i^p b_{ij} x_i x_j - \sum_{k \leq N} k^p b_{kj} x_k x_j
\]
\[
= \frac{1}{2} \sum_{k \leq N} \sum_{i+j=k} \sum_{1 \leq q \leq p-1} \binom{p}{q} i^q j^{p-q} b_{ij} x_i x_j
\]
\[
- \sum_{k \leq N} \sum_{k,j \leq N} k^p b_{kj} x_k x_j.
\]
The desired inequality follows from the hypothesis that \( x \) is nonnegative:
\[
\frac{d}{dt} \sum_{k \leq N} k^p x_k - \sum_{k \leq N} k^p c_k < \frac{1}{2} \sum_{k \leq N} \sum_{i+j=k} \sum_{1 \leq q \leq p-1} \binom{p}{q} i^q j^{p-q} b_{ij} x_i x_j
\]
\[
< \frac{1}{2} \sum_{1 \leq q \leq p-1} \binom{p}{q} \sum_{i,j \leq N} i^q j^{p-q} b_{ij} x_i x_j.
\]

Lemma 4. Let \( b_{ij} < i + j \). Let \( \Sigma_k k^p a_k \) be finite, and \( \Sigma_k k^p c_k \) bounded on bounded intervals, for some positive integer \( p \). Then (1) has the bounded \( p \)-moment property.

Proof. Define
\[
m_1(t) = \int_0^t \sum_k k c_k(\tau) \, d\tau + \sum_k k a_k,
\]
\[
w(t) = \int_0^t m_1(\tau) \, d\tau,
\]
\[
m_2(t) = e^{2w(t)} \left\{ \int_0^t e^{-2w(\tau)} \sum_k k^2 c_k(\tau) \, d\tau + \sum_k k^2 a_k \right\}
\]
\[
\vdots \vdots \vdots
\]
\[
m_p(t) = e^{pw(t)} \left\{ \int_0^t e^{-pw(\tau)} \left( \sum_{1 \leq q \leq p-2} \binom{p}{q} m_{q+1}(\tau) m_{p-q}(\tau) \right. \right.
\]
\[
+ \sum_k k^p c_k(\tau) \right\} \, d\tau + \sum_k k^p a_k \right\}, \quad p > 2.
\]
The \( m_p \) are continuous, because the integrands involved in their definition are bounded. Let \( x \in \bigcup_{b' < b} S_{e}(b', c) \); it will be shown, by induction on \( p \), that \( \Sigma_k k^p x_k < m_p \).

For \( p = 1 \), the right-hand side of (3) is zero, so that
\[
\sum_{k \leq N} k x_k < \int_0^t \sum_{k \leq N} k c_k(\tau) \, d\tau + \sum_{k \leq N} k a_k < m_1(t)
\]
for all \( N \). Letting \( N \) increase yields the desired result. Suppose now that \( \sum k q x_k \leq m_q \) for all \( q < p, \ p > 2 \). Substitution of the bound \( b_{ij} < i + j \) into (3) gives

\[
\frac{d}{dt} \sum_{k<N} k^n x_k < \sum_{1<q<p-1} \binom{p}{q} \sum_{i<N} i^{q+1} x_i \sum_{j<N} j^{p-q} x_j + \sum_{k<N} k^n c_k
\]

Integration of this inequality gives \( \sum_{k<N} k^n x_k(t) < m_p(t) \) for all \( N \). Letting \( N \) increase again yields the desired result.

Proof of Theorem 1. Lemma 4 establishes that (1) has the bounded \( p \)-moment property under the hypotheses of Theorem 1 (since \( i^n + j^n < i + j \)). By Lemma 2 (since \( i^n + j^n < 2i^n j^n \)), (1) therefore admits at least one nonnegative solution on \([0, \infty)\). By definition of the bounded \( p \)-moment property, the \( p \)-moment of any nonnegative solution is bounded on bounded intervals.

References


