PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, October 1980

A GLOBAL EXISTENCE THEOREM FOR SMOLUCHOWSKI'S COAGULATION EQUATIONS

WARREN H. WHITE

ABSTRACT. A countable system of nonlinear ordinary differential equations is shown to admit solutions valid over all positive time.

Consider the following countable system of equations on $[0, \infty)$, $k = 1, 2, \ldots$ [1], [2], [3]:

$$\frac{dx_k}{dt} - c_k = \frac{1}{2} \sum_{i+j=k} b_{ij} x_i x_j - x_k \sum_j b_{kj} x_j, \tag{1a}$$

$$x_k(0) = a_k \geq 0, \tag{1b}$$

where the b_{ij} and c_k are continuous functions of t such that $b_{ij} = b_{ji} \geq 0$ and $c_k \geq 0$. The b_{ij} arising in applications typically increase without bound as i and j increase, so that (1) is beyond the reach of standard results on differential equations in Banach spaces. The case $b_{ij} \leq ij$ has been studied by McLeod [4], [5], who established the existence and uniqueness of a nonnegative ($x_k \geq 0$) initial solution which does not, in general, extend to $t > 1$. The b_{ij} arising in most applications satisfy the more restrictive condition $b_{ij} \leq i + j$, for which case the present note demonstrates the existence of global, well-behaved solutions.

THEOREM 1. *Let $b_{ij} \leq i^\alpha + j^\alpha$ for some $\alpha \in [0, 1]$. Let $\sum_k k^p a_k$ be finite, and $\sum_k k^p c_k$ bounded on bounded intervals, for some integer $p > \alpha$. Then* (1) *admits at least one nonnegative solution on* $[0, \infty)$. *For any nonnegative solution x, $\sum_k k^p x_k$ is bounded on bounded intervals.*

The proof of Theorem 1 begins with a technical definition abstracting conditions necessary to carry through McLeod's local arguments [4] on a global basis. It will be convenient to denote sequences and kernels by the unsubscripted symbols used for their elements (e.g., $(a_k) = a$, $(b_{ij}) = b$), and to write $b' \leq b$ to indicate that $b'_{ij} \leq b_{ij}$ for all i and j. With these conventions, let $S_a(b, c)$ be the (possibly empty) set of nonnegative solutions to (1) on $[0, \infty)$.

DEFINITION. Problem (1) has the *bounded β-moment property* for some $\beta \in [0, \infty)$ if there exists a continuous function m_β: $[0, \infty) \to [0, \infty)$ such that $\sum_k k^\beta x_k \leq m_\beta$ for all $x \in \bigcup_{b' \leq b} S_a(b', c)$.

LEMMA 2. *Let $b_{ij} \leq B i^\alpha j^\alpha$ for some $\alpha \in [0, \infty)$. If* (1) *has the bounded β-moment property for some $\beta > \alpha$, then it admits at least one nonnegative solution on* $[0, \infty)$.

Received by the editors October 29, 1979.
1980 *Mathematics Subject Classification*. Primary 34A10; Secondary 45J05, 70F35.

Proof. The proof of this lemma closely follows the approach taken by McLeod in the proof of his Theorem 2 [**4**].

For each positive integer N, define a truncated kernel $b^{(N)}$ as follows.

$$b_{ij}^{(N)} = \begin{cases} b_{ij} & \text{for } i + j \leq N, \\ 0 & \text{for } i + j > N. \end{cases}$$

Straightforward finite-dimensional arguments show the solution set $S_a(b^{(N)}, c)$ to consist of a single element, $x^{(N)}$. Assuming (1) to have the bounded β-moment property, there exists a continuous function m_β: $[0, \infty) \to [0, \infty)$ such that $m_\beta \geq \sum_k k^\beta x_k^{(N)}$ for each N. Since the $x^{(N)}$ are nonnegative, $x_k^{(N)}$ and $dx_k^{(N)}/dt$ are then bounded on bounded intervals, uniformly for all N:

$$x_k^{(N)} \leq \sum_j j^\beta x_j^{(N)} \leq m_\beta,$$

$$\left|\frac{d}{dt} x_k^{(N)}\right| = \left|\frac{1}{2} \sum_{i+j=k} b_{ij}^{(N)} x_i^{(N)} x_j^{(N)} - x_k^{(N)} \sum_j b_{kj}^{(N)} x_j^{(N)} + c_k\right|$$

$$\leq \frac{B}{2} \sum_i \sum_j i^\beta j^\beta x_i^{(N)} x_j^{(N)} + B k^\beta x_k^{(N)} \sum_j j^\beta x_j^{(N)} + c_k < 2Bm_\beta^2 + c_k.$$

Ascoli's lemma can therefore be applied, repeatedly, to obtain an increasing sequence J of integers such that $\{x_k^{(N)}: N \in J\}$ is uniformly convergent on bounded intervals for each k. In what follows, N will be restricted to values in J.

Let $x_k = \lim_{N\to\infty} x_k^{(N)}$ for each k. To verify equation (1a) in the limit, it suffices to show that $\sum_j b_{kj}^{(N)} x_j^{(N)}$ converges to $\sum_j b_{kj} x_j$ uniformly on bounded intervals. For each k,

$$\left|\sum_j b_{kj} x_j - \sum_j b_{kj}^{(N)} x_j^{(N)}\right|$$

$$\leq \sum_{j \leq N_1} \left|b_{kj} x_j - b_{kj}^{(N)} x_j^{(N)}\right| + \sum_{N_1 < j} \left(b_{kj} x_j + b_{kj}^{(N)} x_j^{(N)}\right). \tag{2}$$

The second term on the right-hand side can be made arbitrarily small by choosing N_1 large:

$$\sum_{N_1 < j} \left(b_{kj} x_j + b_{kj}^{(N)} x_j^{(N)}\right) \leq Bk^\alpha \sum_{N_1 < j} j^\alpha \left(x_j + x_j^{(N)}\right)$$

$$\leq Bk^\alpha \sum_{N_1 < j} N_1^{\alpha-\beta} j^\beta \left(x_j + x_j^{(N)}\right) \leq 2Bk^\alpha m_\beta N_1^{\alpha-\beta}.$$

The first term on the right-hand side of (2) can be made arbitrarily small–for fixed N_1–by choosing N large.

Proposition 3. *Let x be a nonnegative solution to* (1). *Then*

$$\frac{d}{dt} \sum_{k \leq N} k^p x_k - \sum_{k \leq N} k^p c_k \leq \frac{1}{2} \sum_{1 \leq q \leq p-1} \binom{p}{q} \sum_{i \leq N} \sum_{j \leq N} i^q j^{p-q} b_{ij} x_i x_j \tag{3}$$

for all positive integers p and N.

Proof. From equation (1a),

$$\begin{aligned}\frac{d}{dt}\sum_{k\le N}k^p x_k-\sum_{k\le N}k^p c_k&=\sum_{k\le N}k^p\left(\frac{dx_k}{dt}-c_k\right)\\&=\frac{1}{2}\sum_{k\le N}k^p\sum_{i+j=k}b_{ij}x_ix_j-\sum_{k\le N}k^px_k\sum_j b_{kj}x_j\\&=\frac{1}{2}\sum_{k\le N}\sum_{i+j=k}(i+j)^pb_{ij}x_ix_j-\sum_{k\le N}\sum_j k^pb_{kj}x_kx_j\\&=\frac{1}{2}\sum_{k\le N}\sum_{i+j=k}\sum_{1\le q\le p-1}\binom{p}{q}i^qj^{p-q}b_{ij}x_ix_j\\&\quad+\sum_{k\le N}\sum_{i+j=k}i^pb_{ij}x_ix_j-\sum_{k\le N}\sum_j k^pb_{kj}x_kx_j\\&=\frac{1}{2}\sum_{k\le N}\sum_{i+j=k}\sum_{1\le q\le p-1}\binom{p}{q}i^qj^{p-q}b_{ij}x_ix_j\\&\quad-\sum_{k\le N}\sum_{N-k<j}k^pb_{kj}x_kx_j.\end{aligned}$$

The desired inequality follows from the hypothesis that x is nonnegative:

$$\begin{aligned}\frac{d}{dt}\sum_{k\le N}k^p x_k-\sum_{k\le N}k^p c_k&\le\frac{1}{2}\sum_{k\le N}\sum_{i+j=k}\sum_{1\le q\le p-1}\binom{p}{q}i^qj^{p-q}b_{ij}x_ix_j\\&\le\frac{1}{2}\sum_{1\le q\le p-1}\binom{p}{q}\sum_{i\le N}\sum_{j\le N}i^qj^{p-q}b_{ij}x_ix_j.\end{aligned}$$

Lemma 4. *Let $b_{ij}\le i+j$. Let $\sum_k k^pa_k$ be finite, and $\sum_k k^pc_k$ bounded on bounded intervals, for some positive integer p. Then* (1) *has the bounded p-moment property.*

Proof. Define

$$m_1(t)=\int_0^t\sum_k kc_k(\tau)\,d\tau+\sum_k ka_k,\qquad w(t)=\int_0^t m_1(\tau)\,d\tau,$$

$$m_2(t)=e^{2w(t)}\left\{\int_0^t e^{-2w(\tau)}\sum_k k^2c_k(\tau)\,d\tau+\sum_k k^2a_k\right\}$$

$$\vdots\qquad\vdots\qquad\vdots$$

$$m_p(t)=e^{pw(t)}\left\{\int_0^t e^{-pw(t)}\left(\sum_{1\le q\le p-2}\binom{p}{q}m_{q+1}(\tau)m_{p-q}(\tau)+\sum_k k^pc_k(\tau)\right)d\tau+\sum_k k^pa_k\right\},\quad p\ge 2.$$

The m_p are continuous, because the integrands involved in their definition are bounded. Let $x\in\bigcup_{b'\le b}S_a(b',c)$; it will be shown, by induction on p, that $\sum_k k^px_k\le m_p$.

For $p=1$, the right-hand side of (3) is zero, so that

$$\sum_{k\le N}kx_k\le\int_0^t\sum_{k\le N}kc_k(\tau)\,d\tau+\sum_{k\le N}ka_k\le m_1(t)$$

for all N. Letting N increase yields the desired result. Suppose now that $\sum_k k^q x_k \leqslant m_q$ for all $q < p$, $p \geqslant 2$. Substitution of the bound $b_{ij} \leqslant i + j$ into (3) gives

$$\frac{d}{dt} \sum_{k \leqslant N} k^p x_k \leqslant \sum_{1 \leqslant q \leqslant p-1} \binom{p}{q} \sum_{i \leqslant N} i^{q+1} x_i \sum_{j \leqslant N} j^{p-q} x_j + \sum_{k \leqslant N} k^p c_k$$

$$\leqslant p m_1 \sum_{k \leqslant N} k^p x_k + \sum_{1 \leqslant q \leqslant p-2} \binom{p}{q} m_{q+1} m_{p-q} + \sum_k k^p c_k.$$

Integration of this inequality gives $\sum_{k \leqslant N} k^p x_k(t) \leqslant m_p(t)$ for all N. Letting N increase again yields the desired result.

Proof of Theorem 1. Lemma 4 establishes that (1) has the bounded p-moment property under the hypotheses of Theorem 1 (since $i^\alpha + j^\alpha \leqslant i + j$). By Lemma 2 (since $i^\alpha + j^\alpha \leqslant 2 i^\alpha j^\alpha$), (1) therefore admits at least one nonnegative solution on $[0, \infty)$. By definition of the bounded p-moment property, the p-moment of any nonnegative solution is bounded on bounded intervals.

References

1. M. v. Smoluchowski, *Versuch einer mathematischen Theorie der koagulationskinetik kolloider Losungen*, Z. Phys. Chem. **92** (1918), 129–168.

2. R. L. Drake, *A general mathematical survey of the coagulation equation*, Topics in Current Aerosol Research 3 (Part 2), G. M. Hidy and J. R. Brock (eds.), Pergamon, Oxford, 1972, pp. 201–376.

3. H. R. Pruppacher and J. D. Klett, *Microphysics of clouds and precipitation*, Reidel, Dordrecht, 1978.

4. J. B. McLeod, *On an infinite set of nonlinear differential equations*, Quart. J. Math. Oxford Ser. (2) **13** (1962), 119–128.

5. ______, *On an infinite set of nonlinear differential equations*. II, Quart. J. Math. Oxford Ser. (2) **13** (1962), 193–205.

CAPITA, Box 1185, Washington University, St. Louis, Missouri 63130