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MAXIMAL OPERATORS ASSOCIATED TO RADIAL FUNCTIONS IN $L^2(\mathbf{R}^2)$

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ABSTRACT. Stein's result on spherical means imply that for $n \geq 3$ the maximal operator associated to a radial function maps $L^p(\mathbf{R}^n)$ boundedly into itself for $p > n/(n-1)$. In this paper we take a look at the case $p = n = 2$.

1. Introduction. It is a classical result (see e.g. Stein, [**1**, pp. 62–65]) that the maximal operator associated to a radial nonincreasing function in $L^1(\mathbf{R}^n)$ maps $L^p(\mathbf{R}^n)$ boundedly into itself for $1 < p \leq \infty$. By Stein's result on spherical means [**2**], it is possible to remove the monotonicity condition if $n \geq 3$ and $p > n/(n-1)$. On the other hand it was known that if the (not necessarily radial) function is in $L^p(\mathbf{R}^n)$ and has compact support then the maximal operator maps $L^q(\mathbf{R}^n)$ into $L^q(\mathbf{R}^n)$ for $q > p/(p-1)$. This is obtained by using Hölder's inequality and the results for a variant of the Hardy-Littlewood maximal function. In this paper we try to show that for a radial function when $q = 2$, we may take any $p > 1$ as a corollary (Corollary 2) of a more general result. Also the already mentioned result involving monotonicity is shown, in this case, to be a corollary of our theorem.

NOTATIONS. By $\hat{f}$ we denote the Fourier transform of the function $f, \hat{f}(x) = \int_{\mathbf{R}^n} e^{-i\langle x,y\rangle} f(y)\, dy$. C denotes as usual a constant depending on the dimension only which may not be the same at each occurrence.

2. Our aim is to prove the following theorem.

THEOREM. *Let ϕ be a radial function defined on $\mathbf{R}^2$, $\phi \in L^1(\mathbf{R}^2)$ satisfying in addition*

$$\int_{\mathbf{R}^2} |x|^{2(p-1)} |\phi(x)|^2\, dx < \infty$$

for some p, $1 < p \leq 2$. Let $\phi_\varepsilon(x) = \varepsilon^{-2}\phi(\varepsilon^{-1}x)$ and define for $f \in L^2(\mathbf{R}^2)$ and $\varepsilon > 0$, $M_\varepsilon f(x) = \phi_\varepsilon * f(x)$. Consider the maximal operator $M^*f(x) = \sup_{\varepsilon>0} |M_\varepsilon f(x)|$. Then if for $s \geq 1$, $B_s = (\int |x|^{2(s-1)} |\phi(x)|^s\, dx)^{1/s}$,

$$\|M^*f\|_{L^2(\mathbf{R}^2)} \leq C_p(B_1 + B_p)\|f\|_{L^2(\mathbf{R}^2)}$$

where C_p depends on p.

PROOF. Using Hankel's formula we have

$$\hat{\phi}(x) = (2\pi)\int_0^\infty r\eta(r) J_0(|x|r)\, dr$$

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where $J_\nu(z)$ denotes the Bessel function of the first kind of order ν and $\eta(|x|) = \phi(x)$. Using Sonine's first finite integral (see e.g. Watson [3, p. 373]), we obtain

$$J_0(z) = c_\alpha z^{-\alpha} \int_0^1 J_\alpha(zs) s^{\alpha+1}(1 - s^2)^{-\alpha-1}\, ds$$

whenever $-1 < \alpha < 0$. We will choose our α so that $-1/2 < \alpha < 1/2 - 1/p \leqslant 0$. This is possible since $1 < p \leqslant 2$. Therefore

$$\hat{\phi}(x) = 2\pi c_\alpha \int_0^\infty r\eta(r) \int_0^1 \frac{J_\alpha(rs|x|)}{(rs|x|)^\alpha} s^{2\alpha+1}(1 - s^2)^{-\alpha-1}\, ds\, dr. \tag{1}$$

Since $\alpha > -1/2$, we can write (see e.g. Watson [3, p. 48]),

$$J_\alpha(z) = \frac{2\left(\frac{z}{2}\right)^\alpha}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)} \int_0^1 (1 - t^r)^{\alpha - 1/2} \cos zt\, dt$$

so that for $z > 0$ we have the bound

$$\left| \frac{J_\alpha(z)}{z^\alpha} \right| \leqslant 1/2^\alpha \Gamma(\alpha + 1). \tag{2}$$

Hence the integral in (1) is absolutely convergent and we may rewrite it as

$$\hat{\phi}(x) = 2\pi c_\alpha \int_0^\infty \frac{J_\alpha(r|x|)}{(r|x|)^\alpha} \left(\int_0^1 \left(\frac{r}{s}\right)^2 \eta\left(\frac{r}{s}\right) s^{2\alpha+2}(1 - s^2)^{-\alpha-1} \frac{ds}{s} \right) \frac{dr}{r}. \tag{3}$$

Let us denote by $\omega(r)$ the inner integral in (3) multiplied by $2\pi c_\alpha$. Denoting by $M_\varepsilon^\alpha f$ the operator defined by

$$(M_\varepsilon^\alpha f)\hat{}(x) = \frac{J_\alpha(\varepsilon|x|)}{(\varepsilon|x|)^\alpha} \hat{f}(x),$$

we are led to write

$$M_\varepsilon f(x) = \phi_\varepsilon * f(x) = \int_0^\infty M_{\varepsilon r}^\alpha f(x) \omega(r) \frac{dr}{r}.$$

To see that this is indeed true, we observe that each side is in $L^2(\mathbf{R}^2)$ since $\phi \in L^1(\mathbf{R}^2)$,

$$\int_0^\infty |\omega(r)| \frac{dr}{r} = \|\phi\|_{L^1(\mathbf{R})^2}$$

and (2). Taking the Fourier transform of each side acting on the same function $h \in L^2(\mathbf{R}^2)$, it is seen then that these coincide, proving the validity of the equality.

Consider now a smooth radial function ψ having compact support and such that

$$\hat{\psi}(0) = \int_{\mathbf{R}^2} \psi(x) dx = \lim_{z \to 0} \frac{J_\alpha(z)}{z^\alpha} = \frac{1}{2^\alpha \Gamma(\alpha + 1)}.$$

Let $\tilde{M}_\varepsilon f(x) = \psi_\varepsilon * f(x)$ and $\tilde{M}^* f(x) = \sup_{\varepsilon > 0} |\tilde{M}_\varepsilon f(x)|$. It is known that $\tilde{M}^*$ maps $L^2(\mathbf{R}^2)$ boundedly into itself. We have

$$M_\varepsilon f(x) = \int_0^\infty (M_{\varepsilon r}^\alpha f(x) - \tilde{M}_{\varepsilon r} f(x)) \omega(r) \frac{dr}{r} + \int_0^\infty \tilde{M}_{\varepsilon r} f(x) \omega(r) \frac{dr}{r}.$$

The last integral on the right hand side is bounded by $C\tilde{M}^*f(x)$. For the first one we use Schwarz's inequality to obtain

$$\left|\int_0^\infty (M_{\varepsilon r}^\alpha f(x) - \tilde{M}_{\varepsilon r} f(x))\omega(r)\frac{dr}{r}\right| \leqslant g_\alpha f(x)\left(\int_0^\infty |\omega(r)|^2 \frac{dr}{r}\right)^{1/2}$$

where

$$g_\alpha f(x) = \left(\int_0^\infty |M_{\varepsilon r}^\alpha f(x) - \tilde{M}_{\varepsilon r} f(x)|^2 \frac{dr}{r}\right)^{1/2}$$
$$= \left(\int_0^\infty |M_r^\alpha f(x) - \tilde{M}_r f(x)^2 \frac{dr}{r}\right)^{1/2}$$

is the function considered by Stein in [2, p. 2174], where he shows that it maps $L^2(\mathbf{R}^2)$ boundedly into itself for $\alpha > -1/2$. Finally the integral $(\int_0^\infty |\omega(r)|^2 dr/r)^{1/2}$ may be considered, except for a constant factor, as defining the $L^2((0, \infty), dr/r)$ norm of the convolution (in the multiplicative group of $(0, \infty)$ with the measure dr/r) of the functions $r^2\eta(r)$ and the function defined as $r^{2(\alpha+1)}(1 - r^2)^{-\alpha-1}$ for $1 < r < 1$ and 0 elsewhere. Using Young's inequality with $1 = 1/p + 1/q - 1/2$, we obtain

$$\left(\int_0^\infty |\omega(r)|^2 \frac{dr}{r}\right)^{1/2} \leqslant C\left(\int_0^\infty (r^2|\eta(r)|^2)^p \frac{dr}{r}\right)^{1/2}$$
$$\times\left(\int_0^1 \left[(1 - r^2)^{-\alpha-1} r^{2(\alpha+1)}\right]^q \frac{dr}{r}\right)^{1/2}.$$

The first integral is $CB_p = C(\int_{\mathbf{R}^2} |x|^{2(p-1)}|\phi(x)|^p\, dx)^{1/p}$, whereas the second one is convergent by our choice of α. Recall that $-1/2 < \alpha < 1/2 - 1/p = 1/q - 1$. Therefore

$$|M_\varepsilon f(x)| \leqslant C_p\left(B_1 \tilde{M}^* f(x) + B_p g_\alpha f(x)\right).$$

Since the bound is independent of $\varepsilon > 0$, the theorem is proved.

COROLLARY 1. *The result of the theorem is true even if we consider the function ϕ varying with x. In other words suppose that for each $x \in \mathbf{R}^2$ we consider a function $\phi(x, y)$ which is radial in the y variable and such that*

$$\int_{\mathbf{R}^2} |y|^{2(p-1)}|\phi(x, y)|^p\, dy \leqslant K$$

holds uniformly in x for a fixed $p > 1$ and also for $p = 1$. Then the operator

$$\sup_{\varepsilon>0}\left|\int_{\mathbf{R}^2} \phi(x, y) f(x - \varepsilon y)\, dy\right|$$

maps $L^2(\mathbf{R}^2)$ boundedly into itself.

PROOF. We just observe that in the proof of the theorem the bounds B_p and B_1 are used pointwise.

COROLLARY. *If $\phi \in L^p(\mathbf{R}^2)$ for some $p > 1$ and ϕ has compact support and is radial, then the maximal operator*

$$\sup_{\varepsilon>0}\left|\int_{\mathbf{R}^2}\phi(x)f(x-\varepsilon y)\,dy\right|$$

maps $L^2(\mathbf{R}^2)$ boundedly into itself.

PROOF. Clearly ϕ satisfies the hypothesis of the theorem.

REMARK 1. The example given by Stein in [2, p. 2174], showing that the maximal function of the spherical averages is unbounded in $L^2(\mathbf{R}^2)$ can be used to show that the condition $p > 1$ cannot be removed without imposing further conditions on ϕ. For consider for $\alpha > 0$ the functions ϕ^α defined as $\alpha(1-|x|^2)^{\alpha-1}$ if $|x| < 1$ and 0 otherwise. Their Fourier transforms are $2^{\alpha+1}\pi\Gamma(\alpha+1)J_\alpha(|x|)/|x|^\alpha$, showing that $\|\phi^\alpha\|_{L^1(\mathbf{R}^2)} = 2\pi$. Moreover these Fourier transforms converge uniformly in x to the function $2\pi J_0(|x|)$. Now if $f \in \mathbb{S}$ and $\hat{f}$ has compact support, $(\phi_\varepsilon^\alpha * f)\hat{} \in C_0^\infty$ and converges uniformly in $\varepsilon > 0$ and x to $2\pi J_0(\varepsilon|x|)\hat{f}(x)$. So if

$$M_\varepsilon f(x) = \int_{S_1} f(x-\varepsilon y')\,dy'$$

we have $M_\varepsilon f(x) = \lim_{\alpha\to 0^+}\phi_\varepsilon^\alpha * f(x)$ the limit being uniform in $\varepsilon > 0$ and x. Hence

$$\sup_{\varepsilon>0}|M_\varepsilon f(x)| \leqslant \lim_{\alpha\to 0}\ \inf_{\varepsilon\to 0}\sup|\phi_\varepsilon^\alpha * f(x)|$$

and using Fatou's lemma and assuming that the bounds of our theorem depended only on $\|\phi^\alpha\|_{L^2(\mathbf{R}^2)}$ we would get

$$\left\|\sup_{\varepsilon>0}|M_\varepsilon f(x)|\right\|_{L^2(\mathbf{R}^2)} \leqslant C\|f\|_{L^2(\mathbf{R}^2)}$$

whenever $\hat{f} \in C_0^\infty(\mathbf{R}^2)$, but these f's are dense in $L^2_{(\mathbf{R}^2)}$, arriving at a contradicition.

REMARK 2. The classical theorem used when ϕ is a decreasing radial function when applied to the space $L^2(\mathbf{R}^2)$ is a particular case of our theorem. Setting $p = 2$ and $|\phi(x)| = \eta(|x|)$ we see that

$$\begin{aligned}\int_{\mathbf{R}^2}|x|^2|\phi(x)|^2\,dx &= C\int_0^\infty r^3\eta^2(r)\,dr = C\sum_{-\infty}^{+\infty}\int_{2^n}^{2^{n+1}} r^3\eta^2(r)\,dr\\ &\leqslant C\sum_{-\infty}^{+\infty}\eta^2(2^{n+1})2^{4n} \leqslant C\left(\sum_{-\infty}^{+\infty}\eta(2^n)2^{2n}\right)^2\\ &\leqslant C\left(\int_0^\infty r\eta(r)\,dr\right)^2 = C\|\phi\|^2_{L^1(\mathbf{R}^2)}.\end{aligned}$$

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