THE EPIREFLECTIVE HULL OF THE CATEGORY
OF $T_1$ DISPERSED SPACES

V. NEUMANN-LARA AND R. G. WILSON

Abstract. An internal characterization is given of those spaces which can be
embedded in products of $T_1$ dispersed spaces.

A set $X$ with a topology $t$ will be denoted by $(X, t)$.

A subset $Y$ of a topological space $(X, t)$ is said to be autonomous (in $X$) if for all
subspaces $Z$ of $X$ which properly contain $Y$, there are disjoint, nonempty closed
subsets $U$ and $V$ of $Z$ such that $Y \subset U$ and $U \cup V = Z$.

The concept of an autonomous subset was introduced and investigated in [3]. In
particular it was shown there that an autonomous subset of $X$ is closed and that
$a(t) = \{ Y^c : Y = \emptyset \text{ or } Y \text{ is autonomous in } (X, t) \}$ is a topology for $X$ which is
clearly no finer than $t$. In addition, it was proved that the class function $A$ from the
class $TOP$ of all topological spaces to itself defined by $A[(X, t)] = (X, a(t))$ is a
functor on the category of all topological spaces with continuous maps. In the
future, unless confusion may result, we will write $X$ in place of $(X, t)$ and $A(X)$ will
be used to denote $(X, a(t))$. Clearly, any subset of $A(X)$ may be considered a
subset of $X$ and vice versa.

The proofs of the following two lemmas are found in [3].

Lemma 1. Each component and each clopen (open and closed) subset of $X$ is
autonomous.

Lemma 2. If $Y$ is an autonomous subset of $X$ and $W$ is an autonomous subset of $Y$
(with the relative topology), then $W$ is an autonomous subset of $X$.

Theorem 1. For each topological space $X$, $a(a(t)) = a(t)$ and so $A(A(X)) = A(X)$.

Proof. It suffices to show that any closed subset $C$ of $A(X)$ is autonomous in
$A(X)$. Considered as a subset of $X$, $C$ is autonomous. Suppose $Z \supseteq C$, and define
$Z^*$ to be the closure of $Z$ in $A(X)$. Since $C$ is autonomous in $X$ there exist disjoint
nonempty relatively $t$-closed subsets $U$ and $V$ of $Z^*$ such that $C \subset U$ and
$U \cup V = Z^*$. Thus $U$ and $V$ are clopen subsets of $Z^*$ with the relative $t$-topology
and hence by Lemma 1 are autonomous in $Z^*$. Since $Z^*$ is closed in $A(X)$, $Z^*$ with
the relative $t$-topology is autonomous in $X$, and thus by Lemma 2, $U$ and $V$ are autonomous subsets of $X$. Thus $U \cap Z$ and $V \cap Z$ are disjoint relatively $a(t)$-closed subsets of $Z$ such that $U \cap Z \supset C$ and whose union is $Z$. It remains only to show that $V \cap Z \neq \emptyset$. However, $U$ and $V$ are relatively $a(t)$-closed subsets of $Z^*$ which is the $a(t)$-closure of $Z$ and so the result follows.

**Corollary 1.1.** $A(X)$ is totally disconnected if and only if $X$ is totally disconnected.

**Proof.** Since $a(t)$ is no finer than $t$, it follows that if $A(X)$ is totally disconnected so is $X$.

Conversely, if $X$ is totally disconnected then each point of $X$ is a component and hence all singleton subsets of $X$ are autonomous by Lemma 1. It follows immediately that $A(X)$ is $T_1$. If $A(X)$ is not totally disconnected, then there are points of $A(X)$ which are not components, and since it is clear that a proper subset of a component can never be autonomous, it follows that $A(A(X))$ is not $T_1$. This contradicts the theorem.

A topological space $(X, t)$ is said to be **autonomously generated** if $t = a(t)$. Recall that a space is dispersed if every nonempty subspace has an isolated point.

**Theorem 2.** Every $T_1$ zero-dimensional space and every $T_1$ dispersed space is autonomously generated. Also, every autonomously generated $T_1$-space is totally disconnected.

**Proof.** The first and last statements of the theorem are obvious. Now suppose $(X, t)$ is dispersed, $C \subseteq X$ is closed and $Z^* \supset C$. $Z - C$ has an isolated point which must be open and closed in $Z$ since $C$ is closed and $X$ is $T_1$. If $p$ is such a point then $\{ p \}$ and $Z - \{ p \}$ have the required properties.

A space in which every quasi-component is a singleton is said to be **totally separated.** That there exist totally separated spaces which are not autonomously generated is shown by the following example:

Let $X$ be the space of $[4]$, without the dispersion point. $X$ is clearly totally separated. If $Y$ is an odd-numbered row of $X$ then $Y$ is closed in $X$, but there is no clopen subset of $X$ contained in $X - Y$; thus $Y$ is not autonomous in $X$. Indeed, it is not hard to show that $A(A(X)) \simeq Q$, the space of rational numbers.

**Theorem 3.** The category $AG$ of all autonomously generated spaces is epireflective in the category $TOP$ of all topological spaces; furthermore, the functor $A$ is the epireflection.

**Proof.** Let $X$ be a topological space and $Y \in AG$. Then $A(Y) \simeq Y$. But it was shown in [3] that any map $f: X \to Y$ extends to a map $f^*: A(X) \to A(Y)$ in such a way that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
i_x & \downarrow & \downarrow i_y \\
A(X) & \xrightarrow{f^*} & A(Y)
\end{array}
\]
where \( i_x \) denotes the identity map from \( X \) to \( A(X) \).

Since \( A(Y) \approx Y \), the epireflective character of the subcategory \( AG \) follows from the facts that \( i_x \) is clearly an epimorphism and that \( A(X) \in AG \).

**Corollary 3.1.** The category \( AG \) is closed with respect to the taking of products and subspaces.

**Proof.** This is \([1, \text{Theorem 1.2.1}]\).

**Corollary 3.2.** Any subspace of a product of \( T_1 \) dispersed spaces is autonomously generated.

It is clear from Corollary 3.1, that \( \prod_{a \in I}[A(X_a)] \approx A[\prod_{a \in I}A(X_a)] \) for any family \( \{X_a: a \in I\} \) of topological spaces. It follows that the topology of \( A(\prod_{a \in I}X_a) \) is no weaker than the topology of \( \prod_{a \in I}A(X_a) \). We do not know in general whether or not \( A(\prod_{a \in I}X_a) \approx \prod_{a \in I}A(X_a) \).

We denote by \( D(X) \) the quotient space obtained from \( X \) by identifying the points of each component. The quotient map will be denoted by \( d \). It is not hard to show that \( D(X) \) is totally disconnected and hence \( T_1 \).

**Theorem 4.** The category \( AG_1 \) of all autonomously generated \( T_1 \)-spaces is epireflective in \( TOP \) (and in the category \( \mathcal{T}_1 \) of all \( T_1 \)-spaces). Furthermore, the epireflection is \( X \rightarrow A(D(X)) \) and \( A(D(X)) \approx D(A(X)) \).

**Proof.** It is clear that if \( X \in TOP \) or \( \mathcal{T}_1 \), \( Y \in AG_1 \) and \( f: X \rightarrow Y \) is continuous, then there exist maps \( f_d \) and \( f^*_d \) which make the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{d} & D(X) \\
& \downarrow{f} & \downarrow{f_d} \\
& A(D(X)) & \xrightarrow{i_x} A(D(X))
\end{array}
\]

Since \( D(X) \) is totally disconnected, it follows that \( A(D(X)) \in AG_1 \) (Corollary 1.1). Also, \( i_x \circ d \) is clearly an epimorphism, and the result follows.

To show that \( A(D(X)) \) and \( D(A(X)) \) are homeomorphic, we note that if \( Y \in AG_1 \) and \( g: X \rightarrow Y \) is continuous, there exist maps \( g^* \) and \( g_d^* \) which make the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{i_x} & A(X) \\
& \downarrow{g} & \downarrow{g_d^*} \\
& A(X) & \xrightarrow{d} D(A(X))
\end{array}
\]

The result will follow from the unicity of the epireflective object if we can show that \( D(A(X)) \in AG_1 \); or more simply, that \( D(Y) \in AG_1 \) whenever \( Y \in AG \). To this end, let \( C \) be a closed subset of \( D(Y) \). Since \( D(Y) \) is totally disconnected and
hence \( T_1 \), it suffices to show that \( C \) is autonomous. Suppose \( Z \supseteq C \). Since \( d \) is onto, it follows that \( d^{-1}[Z] \supseteq d^{-1}[C] \) and since \( d^{-1}[C] \) is closed it is autonomous in \( Y \). Thus there are nonempty, relatively closed subsets \( U \) and \( V \) of \( d^{-1}[Z] \) such that \( U \cup V = d^{-1}[Z] \) and \( d^{-1}[C] \subseteq U \). Since no proper subset of a component of \( Y \) is autonomous in \( Y \), it follows that the relative topology on each component of \( Y \) is indiscrete. Now there exist closed subsets \( U^* \) and \( V^* \) of \( Y \) such that \( U = U^* \cap d^{-1}[Z] \) and \( V = V^* \cap d^{-1}[Z] \) and so if \( x \in U \) and \( y \in V \) then \( x \in U^* - V^* \) and \( y \in V^* - U^* \). Thus \( x \) and \( y \) belong to different components of \( Y \) and so \( d(x) \neq d(y) \). It follows that \( d[U] \cap d[V] = \emptyset \). Since \( U^* \) and \( V^* \) are closed in \( Y \), they are the union of components of \( Y \) and hence \( d^{-1}[d[U^*]] = U^* \) and \( d^{-1}[d[V^*]] = V^* \) and so \( d[U^*] \) and \( d[V^*] \) are closed in \( D(Y) \); clearly, \( d[U] = d[U^*] \cap Z \) and \( d[V] = d[V^*] \cap Z \) and so \( d[U^*] \cap Z \) and \( d[V^*] \cap Z \) are disjoint relatively closed subsets of \( Z \) whose union is \( Z \) and such that \( C \subseteq d[U^*] \cap Z \).

The following easy lemma is left to the reader:

**Lemma 3.** If \( C \) is autonomous in \( X \) and \( Y \subseteq X \), then either \( C \cap Y = \emptyset \) or \( C \cap Y \) is autonomous in \( Y \).

We are now in a position to prove the main result of this article, namely that the autonomously generated \( T_1 \)-spaces form the epireflective hull in \( \mathcal{T}_1 \) or in TOP of the category of \( T_1 \) dispersed spaces.

**Theorem 5.** A topological space \( X \) is an autonomously generated \( T_1 \)-space if and only if it can be embedded in a product of \( T_1 \) dispersed spaces.

**Proof.** Suppose that \( X \) is an autonomously generated \( T_1 \)-space and let \( C \subseteq X \) be closed. Since \( C \) is autonomous in \( X \), there is an open and closed subset of \( X \) contained in \( X - C \). Let \( \mathcal{G}_1 \) be a maximal family of disjoint clopen subsets of \( X \) contained in \( X - C \). For each ordinal \( \alpha \), having selected \( \mathcal{G}_\beta \) for each \( \beta < \alpha \) and supposing that \( X - \bigcup_{\beta < \alpha} (\bigcup \mathcal{G}_\beta) \supseteq C \), we select \( \mathcal{G}_\alpha \) as follows: \( \mathcal{G}_\alpha \) is a maximal family of disjoint relatively clopen subsets of \( X - \bigcup_{\beta < \alpha} (\bigcup \mathcal{G}_\beta) \) which are disjoint from \( C \). The existence of such a family is guaranteed by Lemma 3. Let \( \sigma \) be the first ordinal for which \( X - \bigcup_{\beta < \sigma} (\bigcup \mathcal{G}_\beta) = C \) and let \( \mathcal{G}_\sigma = C \). Then the family \( X_\sigma = \{ F : F \in \mathcal{G}_\sigma \text{ some } \alpha < \sigma \} \) is a partition of \( X \) and it is easy to see that each member of this partition is a closed subset of \( X \). If \( X_\sigma \) is now given the quotient topology and \( q_\sigma : X \to X_\sigma \) denotes the quotient map, then it is clear that \( X_\sigma \) becomes a \( T_1 \)-space and that if \( p \in X - C \), then \( q_\sigma(p) \notin q_\sigma[C] \); in other words, \( q_\sigma \) separates \( C \) from any point \( p \in X - C \). The necessity follows from [5, Theorem 8.16] if we can show that \( X_\sigma \) is dispersed. Let \( B \subseteq X_\sigma \) and \( \alpha = \min\{ \beta : \{ F \} \in B \text{ for some } F \in \mathcal{G}_\beta \} \). Now fix \( F_0 \in \mathcal{G}_\alpha \) such that \( \{ F_0 \} \in B \). It suffices to show that \( \{ F_0 \} \) is an isolated point of \( q_\sigma[X - \bigcup_{\gamma < \alpha} (\bigcup \mathcal{G}_\gamma)] \) since this latter set clearly contains \( B \). Since \( \{ F_0 \} \) is closed in \( X_\alpha \), it is required to show that \( \{ F_0 \} \) is open in \( q_\sigma[X - \bigcup_{\gamma < \alpha} (\bigcup \mathcal{G}_\gamma)] \); thus we need to find an open set \( U \) in \( X_\alpha \) such that \( \{ F_0 \} = U \cap q_\sigma[X - \bigcup_{\gamma < \alpha} (\bigcup \mathcal{G}_\gamma)] \). We take \( U = q_\sigma[\bigcup_{\gamma < \alpha} (\bigcup \mathcal{G}_\gamma) \cup F_0] \) which is open in \( X_\alpha \) since \( q_\sigma^{-1}[U] = \bigcup_{\gamma < \alpha} (\bigcup \mathcal{G}_\gamma) \cup F_0 \) which is open in \( X \), and clearly has the desired property.
The theorem now follows from Corollary 3.2.

The space $X_c$ constructed in the above theorem is not in general uniquely determined but will depend on the selection of the families $\mathcal{F}_a$. If $X$ is a $T_2$-space, $X_c$ may or may not be Hausdorff for a fixed $C$—in fact, if $X$ is nonregular, at least one of the spaces $X_c$ will not be Hausdorff. However, we do not know whether or not every autonomously generated $T_2$-space can be embedded in a product of $T_2$ dispersed spaces.

We now give another characterization of autonomously generated $T_1$-spaces.

**Theorem 6.** A $T_1$-space $X$ is autonomously generated if and only if for each closed set $C \subseteq X$, there is a $T_1$ dispersed space $Y$ and a continuous surjection $f: X \to Y$ such that $C = f^{-1}[y]$ for some $y \in Y$.

**Proof.** The necessity follows from the construction described in Theorem 5.

For the sufficiency, let $C \subseteq X$ be closed, $Y$ be a $T_1$ dispersed space, $f: X \to Y$ continuous and onto such that $C = f^{-1}[y]$ for some $y \in Y$. If $Z \supsetneq C$, then it follows that $f[Z] \supsetneq f[C]$. But $Y$ is $T_1$ and dispersed and hence is totally disconnected; thus $f[C] = \{y\}$ is autonomous. Thus there is a relatively clopen subset $U$ of $f[Z]$ disjoint from $f[C]$. Then $f^{-1}[U]$ is relatively open and closed in $f^{-1}[f[Z]]$ and so $f^{-1}[U] \cap Z$ is relatively open and closed in $Z$, is disjoint from $C$ and is nonempty.

As a final remark, we note that the category of all autonomously generated $T_1$-spaces cannot be simply generated either in $\mathbf{TOP}$ or $\mathbf{S}_1$, since the spaces $Q_n$ constructed in [2] are all $T_2$ and dispersed.

**Bibliography**