PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, October 1980

THE EPIREFLECTIVE HULL OF THE CATEGORY OF T_1 DISPERSED SPACES

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ABSTRACT. An internal characterization is given of those spaces which can be embedded in products of T_1 dispersed spaces.

A set X with a topology t will be denoted by (X, t).

A subset Y of a topological space (X, t) is said to be *autonomous* (in X) if for all subspaces Z of X which properly contain Y, there are disjoint, nonempty closed subsets U and V of Z such that $Y \subset U$ and $U \cup V = Z$.

The concept of an autonomous subset was introduced and investigated in [3]. In particular it was shown there that an autonomous subset of X is closed and that $a(t) = \{Y^c\colon Y = \varnothing \text{ or } Y \text{ is autonomous in } (X, t)\}$ is a topology for X which is clearly no finer than t. In addition, it was proved that the class function A from the class TOP of all topological spaces to itself defined by $A[(X, t)] = (X, a(t))$ is a functor on the category of all topological spaces with continuous maps. In the future, unless confusion may result, we will write X in place of (X, t) and $A(X)$ will be used to denote $(X, a(t))$. Clearly, any subset of $A(X)$ may be considered a subset of X and vice versa.

The proofs of the following two lemmas are found in [3].

LEMMA 1. *Each component and each clopen (open and closed) subset of X is autonomous.*

LEMMA 2. *If Y is an autonomous subset of X and W is an autonomous subset of Y (with the relative topology), then W is an autonomous subset of X.*

THEOREM 1. *For each topological space X, $a(a(t)) = a(t)$ and so $A(A(X)) = A(X)$.*

PROOF. It suffices to show that any closed subset C of $A(X)$ is autonomous in $A(X)$. Considered as a subset of X, C is autonomous. Suppose $Z \supsetneq C$, and define Z^* to be the closure of Z in $A(X)$. Since C is autonomous in X there exist disjoint nonempty relatively t-closed subsets U and V of Z^* such that $C \subset U$ and $U \cup V = Z^*$. Thus U and V are clopen subsets of Z^* with the relative t-topology and hence by Lemma 1 are autonomous in Z^*. Since Z^* is closed in $A(X)$, Z^* with

Received by the editors July 17, 1979 and, in revised form, October 14, 1979; presented to the Society, January 5, 1980.

AMS (MOS) subject classifications (1970). Primary 54F65.

Key words and phrases. Dispersed space, epireflection, autonomous subset.

[1]This author gratefully acknowledges the support of the Consejo Nacional de Ciencia y Tecnología, grant PNCB 1626.

the relative t-topology is autonomous in X, and thus by Lemma 2, U and V are autonomous subsets of X. Thus $U \cap Z$ and $V \cap Z$ are disjoint relatively $a(t)$-closed subsets of Z such that $U \cap Z \supset C$ and whose union is Z. It remains only to show that $V \cap Z \neq \varnothing$. However, U and V are relatively $a(t)$-closed subsets of Z^* which is the $a(t)$-closure of Z and so the result follows.

Corollary 1.1. *$A(X)$ is totally disconnected if and only if X is totally disconnected.*

Proof. Since $a(t)$ is no finer than t, it follows that if $A(X)$ is totally disconnected so is X.

Conversely, if X is totally disconnected then each point of X is a component and hence all singleton subsets of X are autonomous by Lemma 1. It follows immediately that $A(X)$ is T_1. If $A(X)$ is not totally disconnected, then there are points of $A(X)$ which are not components, and since it is clear that a proper subset of a component can never be autonomous, it follows that $A(A(X))$ is not T_1. This contradicts the theorem.

A topological space (X, t) is said to be *autonomously generated* if $t = a(t)$. Recall that a space is *dispersed* if every nonempty subspace has an isolated point.

Theorem 2. *Every T_1 zero-dimensional space and every T_1 dispersed space is autonomously generated. Also, every autonomously generated T_1-space is totally disconnected.*

Proof. The first and last statements of the theorem are obvious. Now suppose (X, t) is dispersed, $C \subset X$ is closed and $Z \underset{\neq}{\supset} C$. $Z - C$ has an isolated point which must be open and closed in Z since C is closed and X is T_1. If p is such a point then $\{p\}$ and $Z - \{p\}$ have the required properties.

A space in which every quasi-component is a singleton is said to be *totally separated*. That there exist totally separated spaces which are not autonomously generated is shown by the following example:

Let X be the space of [4], without the dispersion point. X is clearly totally separated. If Y is an odd-numbered row of X then Y is closed in X, but there is no clopen subset of X contained in $X - Y$; thus Y is not autonomous in X. Indeed, it is not hard to show that $A(X) \simeq Q$, the space of rational numbers.

Theorem 3. *The category AG of all autonomously generated spaces is epireflective in the category TOP of all topological spaces; furthermore, the functor A is the epireflection.*

Proof. Let X be a topological space and $Y \in$ AG. Then $A(Y) \simeq Y$. But it was shown in [3] that any map $f\colon X \to Y$ extends to a map $f^*\colon A(X) \to A(Y)$ in such a way that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_x\downarrow & & \downarrow i_y \\ A(X) & \xrightarrow[f^*]{} & A(Y) \end{array}$$

where i_x denotes the identity map from X to $A(X)$.

Since $A(Y) \cong Y$, the epireflective character of the subcategory AG follows from the facts that i_x is clearly an epimorphism and that $A(X) \in$ AG.

COROLLARY 3.1. *The category AG is closed with respect to the taking of products and subspaces.*

PROOF. This is [**1**, Theorem 1.2.1].

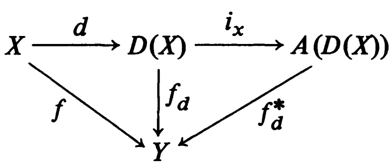
COROLLARY 3.2. *Any subspace of a product of T_1 dispersed spaces is autonomously generated.*

It is clear from Corollary 3.1, that $\Pi_{\alpha\in I}[A(X_\alpha)] \cong A[\Pi_{\alpha\in I}A(X_\alpha)]$ for any family $\{X_\alpha \colon \alpha \in I\}$ of topological spaces. It follows that the topology of $A(\Pi_{\alpha\in I}X_\alpha)$ is no weaker than the topology of $\Pi_{\alpha\in I}A(X_\alpha)$. We do not know in general whether or not $A(\Pi_{\alpha\in I}X_\alpha) \cong \Pi_{\alpha\in I}A(X_\alpha)$.

We denote by $D(X)$ the quotient space obtained from X by identifying the points of each component. The quotient map will be denoted by d. It is not hard to show that $D(X)$ is totally disconnected and hence T_1.

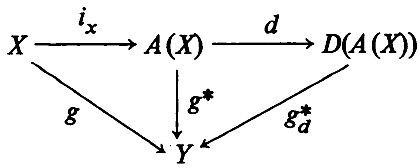
THEOREM 4. *The category AG_1 of all autonomously generated T_1-spaces is epireflective in TOP (and in the category $\mathfrak{T}_1$ of all T_1-spaces). Furthermore, the epireflection is $X \to A(D(X))$ and $A(D(X)) \cong D(A(X))$.*

PROOF. It is clear that if $X \in$ TOP or $\mathfrak{T}_1$, $Y \in \mathrm{AG}_1$ and $f\colon X \to Y$ is continuous, then there exist maps f_d and f_d^* which make the following diagram commutative:



Since $D(X)$ is totally disconnected, it follows that $A(D(X)) \in \mathrm{AG}_1$ (Corollary 1.1). Also, $i_x \circ d$ is clearly an epimorphism, and the result follows.

To show that $A(D(X))$ and $D(A(X))$ are homeomorphic, we note that if $Y \in \mathrm{AG}_1$ and $g\colon X \to Y$ is continuous, there exist maps g^* and g_d^* which make the following diagram commutative:



The result will follow from the unicity of the epireflective object if we can show that $D(A(X)) \in \mathrm{AG}_1$; or more simply, that $D(Y) \in \mathrm{AG}_1$ whenever $Y \in$ AG. To this end, let C be a closed subset of $D(Y)$. Since $D(Y)$ is totally disconnected and hence T_1, it suffices to show that C is autonomous. Suppose $Z \supsetneq C$. Since d is onto, it follows that $d^{-1}[Z] \supsetneq d^{-1}[C]$ and since $d^{-1}[C]$ is closed it is autonomous in Y. Thus there are nonempty, relatively closed subsets U and V of $d^{-1}[Z]$ such that $U \cup V = d^{-1}[Z]$ and $d^{-1}[C] \subset U$. Since no proper subset of a component of Y is autonomous in Y, it follows that the relative topology on each component of Y is indiscrete. Now there exist closed subsets U^* and V^* of Y such that $U = U^* \cap d^{-1}[Z]$ and $V = V^* \cap d^{-1}[Z]$ and so if $x \in U$ and $y \in V$ then $x \in U^* - V^*$ and $y \in V^* - U^*$. Thus x and y belong to different components of Y and so $d(x) \neq d(y)$. It follows that $d[U] \cap d[V] = \varnothing$. Since U^* and V^* are closed in Y, they are the union of components of Y and hence $d^{-1}[d[U^*]] = U^*$ and $d^{-1}[d[V^*]] = V^*$ and so $d[U^*]$ and $d[V^*]$ are closed in $D(Y)$; clearly, $d[U] = d[U^*] \cap Z$ and $d[V] = d[V^*] \cap Z$ and so $d[U^*] \cap Z$ and $d[V^*] \cap Z$ are disjoint relatively closed subsets of Z whose union is Z and such that $C \subset d[U^*] \cap Z$.

The following easy lemma is left to the reader:

LEMMA 3. *If C is autonomous in X and $Y \subset X$, then either $C \cap Y = \varnothing$ or $C \cap Y$ is autonomous in Y.*

We are now in a position to prove the main result of this article, namely that the autonomously generated T_1-spaces form the epireflective hull in $\mathfrak{T}_1$ or in TOP of the category of T_1 dispersed spaces.

THEOREM 5. *A topological space X is an autonomously generated T_1-space if and only if it can be embedded in a product of T_1 dispersed spaces.*

PROOF. Suppose that X is an autonomously generated T_1-space and let $C \subsetneq X$ be closed. Since C is autonomous in X, there is an open and closed subset of X contained in $X - C$. Let $\mathfrak{F}_1$ be a maximal family of disjoint clopen subsets of X contained in $X - C$. For each ordinal α, having selected $\mathfrak{F}_\beta$ for each $\beta < \alpha$ and supposing that $X - \bigcup_{\beta<\alpha}(\bigcup \mathfrak{F}_\beta) \supsetneq C$, we select $\mathfrak{F}_\alpha$ as follows: $\mathfrak{F}_\alpha$ is a maximal family of disjoint relatively clopen subsets of $X - \bigcup_{\beta<\alpha}(\bigcup \mathfrak{F}_\beta)$ which are disjoint from C. The existence of such a family is guaranteed by Lemma 3. Let σ be the first ordinal for which $X - \bigcup_{\beta<\sigma}(\bigcup \mathfrak{F}_\beta) = C$ and let $\mathfrak{F}_\sigma = C$. Then the family $X_c = \{F\colon F \in \mathfrak{F}_\alpha \text{ some } \alpha \leq \sigma\}$ is a partition of X and it is easy to see that each member of this partition is a closed subset of X. If X_c is now given the quotient topology and $q_c\colon X \to X_c$ denotes the quotient map, then it is clear that X_c becomes a T_1-space and that if $p \in X - C$, then $q_c(p) \notin \overline{q_c[C]}$; in other words, q_c separates C from any point $p \in X - C$. The necessity follows from [5, Theorem 8.16] if we can show that X_c is dispersed. Let $B \subset X_c$ and $\alpha = \min\{\beta\colon \{F\} \in B$ for some $F \in \mathfrak{F}_\beta\}$. Now fix $F_0 \in \mathfrak{F}_\alpha$ such that $\{F_0\} \in B$. It suffices to show that $\{F_0\}$ is an isolated point of $q_c[X - \bigcup_{\gamma<\alpha}(\bigcup \mathfrak{F}_\gamma)]$ since this latter set clearly contains B. Since $\{F_0\}$ is closed in X_c, it is required to show that $\{F_0\}$ is open in $q_c[X - \bigcup_{\gamma<\alpha}(\bigcup \mathfrak{F}_\gamma)]$; thus we need to find an open set U in X_c such that $\{F_0\} = U \cap q_c[X - \bigcup_{\gamma<\alpha}(\bigcup \mathfrak{F}_\gamma)]$. We take $U = q_c[\bigcup_{\gamma<\alpha}(\bigcup \mathfrak{F}_\gamma) \cup F_0]$ which is open in X_c since $q_c^{-1}[U] = \bigcup_{\gamma<\alpha}(\bigcup \mathfrak{F}_\gamma) \cup F_0$ which is open in X, and clearly has the desired property.

The theorem now follows from Corollary 3.2.

The space X_c constructed in the above theorem is not in general uniquely determined but will depend on the selection of the families $\mathfrak{F}_\alpha$. If X is a T_2-space, X_c may or may not be Hausdorff for a fixed C–in fact, if X is nonregular, at least one of the spaces X_c will not be Hausdorff. However, we do not know whether or not every autonomously generated T_2-space can be embedded in a product of T_2 dispersed spaces.

We now give another characterization of autonomously generated T_1-spaces.

THEOREM 6. *A T_1-space X is autonomously generated if and only if for each closed set $C \subset X$, there is a T_1 dispersed space Y and a continuous surjection $f: X \to Y$ such that $C = f^{-1}[y]$ for some $y \in Y$.*

PROOF. The necessity follows from the construction described in Theorem 5.

For the sufficiency, let $C \subset X$ be closed, Y be a T_1 dispersed space, $f: X \to Y$ continuous and onto such that $C = f^{-1}[y]$ for some $y \in Y$. If $Z \supsetneq C$, then it follows that $f[Z] \supsetneq f[C]$. But Y is T_1 and dispersed and hence is totally disconnected; thus $f[C] = \{y\}$ is autonomous. Thus there is a relatively clopen subset U of $f[Z]$ disjoint from $f[C]$. Then $f^{-1}[U]$ is relatively open and closed in $f^{-1}[f[Z]]$ and so $f^{-1}[U] \cap Z$ is relatively open and closed in Z, is disjoint from C and is nonempty.

As a final remark, we note that the category of all autonomously generated T_1-spaces cannot be simply generated either in TOP or $\mathfrak{T}_1$, since the spaces Q_α constructed in [2] are all T_2 and dispersed.

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