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HOMOTOPY RIGIDITY FOR GRASSMANNIANS

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ABSTRACT. Two n-dimensional unitary representations which differ by complex conjugation or tensoring with a character induce topologically equivalent actions on the Grassmann manifold of complex m-planes in n-space. This paper shows under modest dimension hypotheses that only such projectively equivalent linear representations of compact connected Lie groups can give topologically conjugate actions.

A linear representation α: $G \to U(N)$ induces a G-action on the Grassmann manifold $G_m(C^N) = U(N)/U(m) \times U(N-m)$ of m-planes in complex N-space. If $\bar{\alpha}$ is the conjugate of α and χ is any character of G, it is obvious that $\alpha \otimes \chi$ and $\bar{\alpha} \otimes \chi$ give rise to the same actions. The purpose of this paper is to show for most values of m and N that only such obviously equivalent linear actions give topologically conjugate actions on $G_m(C^N)$. The main result is as follows.

THEOREM 1. *Let α and β be N-dimensional unitary representations of a compact connected Lie group G. Suppose that there is a homotopy equivalence f: $G_m(C^N) \to G_m(C^N)$ $(N > m)$ which is a G-map between $G_m(C^N)$ with the two linear actions α and β. Further suppose that either*

(a) $N \neq 2m$, *or*

(b) $m \leq 3$.

Then there is a character χ: $G \to S^1$ so that β is equivalent as a linear representation to either $\alpha \otimes \chi$ or $\bar{\alpha} \otimes \chi$.

In the language of Arunas Liulevicius ([1], [2], [3], [4]), this theorem says that compact connected Lie group actions on $G_m(C^N)$ satisfy the property of homotopy rigidity. Wu-Yi Hsiang observed that such results could sometimes be proved efficiently by the use of equivariant cohomology (see e.g. [5]). We shall use this technique to prove Theorem 1. The author would like to express his appreciation to Wu-Yi Hsiang, Arunas Liulevicius, and Birgit Speh for helpful comments.

Because equivariant cohomology is invariant under G-maps which are homotopy equivalences, our strategy for proving Theorem 1 is quite straightforward. We shall simply show that two linear actions which are not obviously equivalent will in fact have nonisomorphic equivariant cohomologies.

The dimension restrictions in Theorem 1 arise completely from uncertainty about the group of automorphisms of the integral cohomology of the Grassman-

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nian in this case. It is a very viable conjecture that only the obvious automorphisms occur.

In order to study the equivariant cohomology, we shall recall some results about the cohomology of the Grassmannian (see e.g. [6]). As is well known, the cohomology of the infinite Grassmannian of m-planes may be identified with the algebra of symmetric functions in m-variables $t_1, t_2, \ldots, t_m$. The elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_m$ are then the Chern classes of the canonical m-plane bundle. Define classes $h_k \in H^{2k}(BU(m); Z) \subseteq Z[t_1, \ldots, t_m]$ inductively by

$$h_0 = 1 \qquad (h_i = 0 \text{ for } i < 0),$$

$$h_{n+m} = \sum_{j=1}^{m} (-1)^{j-1} \sigma_j h_{n+m-j}. \tag{$*$}$$

Let $\langle h_{N-m+1}, h_{N-m+2}, \ldots, h_N \rangle$ be the ideal in $Z[h_1, \ldots, h_m] = Z[\sigma_1, \ldots, \sigma_m]$ generated by $h_{N-m+1}, \ldots, h_N$.

THEOREM 2. (1) $H^*(G_m(C^N); Z) = Z[h_1, \ldots, h_m]/\langle h_{N-m+1}, \ldots, h_N \rangle$.

(2) *A module basis for* $H^*(G_m(C^N); Z)$ *is given by all m-fold products*

$$h_{i_1} h_{i_2} \cdots h_{i_m} \quad \textit{with } 0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m \leqslant N - m.$$

The h_k are the so-called "complete symmetric functions." They may be identified (up to sign) with the Chern classes of the canonical $N - m$-plane bundle. Their generating function $H = \sum_{k=0}^{\infty} h_k$ is given by

$$\prod_{j=1}^{m} \left(\frac{1}{1 - t_j} \right) = 1 \Big/ \left(\sum_{j=0}^{m} (-1)^j \sigma_j \right),$$

and hence h_k can also be described as the unique degree k symmetric polynomial each of whose monomials $t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m}$ has coefficient 1.

For compact connected Lie groups, similarity of linear respresentations is entirely detected by restriction to a maximal torus. Also any character of the maximal torus which is invariant under the Weyl group action will be the restriction of a character of the group. Hence it suffices to prove Theorem 1 when G is a torus T. Let $R = H^*(BT; Z) = Z[x_1, \ldots, x_p]$ where p is the rank of the torus. Note that by transgression, the weights of a representation $\alpha\colon T \to U(N)$ may be viewed as two dimensional elements of R.

Recall that the equivariant cohomology $H_T^*(X)$ of a T-space X is defined to be the ordinary cohomology of the space X_T formed by the Borel construction. More precisely, X_T is the total space of the bundle

$$\begin{array}{c} X \to X_T = ET \times_T X \\ \downarrow \\ BT \end{array}$$

which is associated to the universal T-bundle by the T-action on X.

Now in the case where $X = G_m(C^N)$ with a linear T-action, the canonical bundles γ^m and γ^{N-m} are equivariant bundles. Hence the Borel construction may be applied to them to give bundles $(\gamma^m)_T$ and $(\gamma^{N-m})_T$ over X_T. Inasmuch as the

Chern classes $c_i(\gamma^m) = \sigma_i$ generate $H^*(X; Z)$, we see that the Chern classes of $(\gamma^m)_T$ provide explicit lifts of $H^*(X; Z)$ to $H_T^*(X; Z)$. Hence the fiber of the Borel fibration is totally nonhomologous to zero. Also for the trivial bundle of dimension N with nontrivial torus action given by α, it is easy to see that $c_k(\xi_T)$ is the kth elementary symmetric function ρ_k of the weights w_i of α. As $(\gamma^m)_T \oplus (\gamma^{N-m})_T = \xi_T$, we may use the above observations to obtain the following (see [6] for more detail).

THEOREM 3. *Let T act on $G_m(C^N)$ by a linear representation α with weights w_i. Then*

$$H_T^*(G_m(C^N); Z) = R[h_1, \ldots, h_m]/\langle f_{N-m+1}, \ldots, f_N\rangle$$

where

$$f_m = \sum_{j=0}^{M} (-1)^j \rho_j h_{m-j}$$

and ρ_j is the jth elementary symmetric function of the weights w_i.

It is worth noticing that the classes $h_k \in H_T^*(G_m(C^N); Z)$ are defined by the recursion formulas (*) where the σ_i are now taken to be the Chern classes $c_i((\gamma^m)_T)$. The bundle γ^m with action $\alpha \otimes \chi$ is T-isomorphic to the tensor product of γ^m with action α and a 1-dimensional trivial bundle with action χ. Let w be the weight of χ. Then replacement of α by $\alpha \otimes \chi$ replaces the elementary symmetric function σ_k of $t_1, \ldots, t_m$ by the corresponding symmetric function of $t_1 + w, \ldots, t_m + w$.

By restriction to the fiber in the Borel bundle, it is clear that any isomorphism of equivariant cohomologies which is induced by a T-map will in turn induce an automorphism of $H^*(G_m(C^N); Z)$. The following has been shown by Brewster [9] (see also [8] and [7] for special cases with simpler proofs).

THEOREM 4. *Suppose $N \geqslant 3$. If $N \neq 2m$, the only nontrivial automorphism of $H^*(G_m(C^N); Z)$ is the involution induced by conjugation c of complex numbers. Explicitly $c^*(h_k) = (-1)^k h_k$.*

The above result is a special case of [2, Conjecture 5].

CONJECTURE. If $N = 2m$, $m > 1$ then the group of automorphisms is $Z/2 \times Z/2$ generated by conjugation and the map d taking an m-plane to the m-plane orthogonal to it. Explicitly $d^*(h_m) = (-1)^m \sigma_m$. The cases $(N, m) = (4, 2)$ or $(6, 3)$ are easy to verify. Consequently:

LEMMA 5. *With the dimension restriction of Theorem 1, all automorphisms of $H^*(G_m(C^N); Z)$ are induced by maps which are equivariant between $G_m(C^N)$ with action α and $G_m(C^N)$ with action either α or $\bar{\alpha}$.*

In proving Theorem 1, we will also establish homotopy rigidity for any value of $N = 2m$ where Lemma 5 remains valid.

It is also helpful to define a filtration of $R[h_1, \ldots, h_m]$. Let J_k be the ideal

generated by the elements of R having degree greater than or equal to k. (Here the degree of an element is one half its dimension as a cohomology class; i.e. $\deg x_i = 1$.) Clearly $J_r \cdot J_s = J_{r+s}$ and $\bigcap_{k=0}^{\infty} J_k = \{0\}$. Also define S to be the linear span (over R) of all singletons h_k. Notice that the generators of the ideal of relations in H_T^* are elements of S. The following is the essential technical lemma required to prove Theorem 1. We will use the convention $h_x = 0$ if $x < 0$.

LEMMA 6. *Suppose Φ is a graded endomorphism of the algebra $R[h_1, \ldots, h_m]$ satisfying*

$$\Phi(\sigma_i) = \sigma_i + A_i \quad \textit{modulo } J_{k+1} \qquad (k \geqslant 1)$$

where $A_i \in J_k$ for $1 \leqslant i \leqslant m$.

Define elements $P_{r,j}$ by

$$P_{r,j} = \left[\sum_{i=1}^{m} (-1)^{i-1} A_i h_{j-i} \right] h_{r-j}.$$

Then

$$\Phi(h_r) = h_r + \sum_{j=1}^{r} P_{r,j} \quad \textit{modulo } J_{k+1}.$$

PROOF. The proof is by induction on r. The cases $r \leqslant 1$ are trivial. Assume the truth of the lemma for $r < s$. Since $h_s = \sum_{i=1}^{m} (-1)^{i-1} \sigma_i h_{s-i}$, we note that the $P_{s,j}$ satisfy the recurrence relation

$$P_{s,j} = \sum_{i=1}^{m} (-1)^{i-1} \sigma_i P_{s-i,j} \quad \text{for } s > j$$

with the convention $P_{s,j} = 0$ if $s < j$. Now

$$\begin{aligned}
\Phi(h_s) &= \sum_{i=1}^{m} (-1)^{i-1} \Phi(\sigma_i) \Phi(h_{s-i}) \\
&= \sum_{i=1}^{m} (-1)^{i-1} (\sigma_i + A_i) \left(h_{s-i} + \sum_{j=1}^{s-i} P_{s-i,j} \right) \quad \text{modulo } J_{k+1} \\
&= \sum_{i=1}^{m} (-1)^{i-1} \sigma_i h_{s-i} + \sum_{i=1}^{m} (-1)^{i-1} A_i h_{s-i} \\
&\quad + \sum_{i=1}^{m} \sum_{j=1}^{s-i} (-1)^{i-1} \sigma_i P_{s-i,j} \quad \text{modulo } J_{k+1}.
\end{aligned}$$

But

$$\sum_{i=1}^{m} \sum_{j=1}^{s-i} (-1)^{i-1} \sigma_i P_{s-i,j} = \sum_{j=1}^{s-1} \sum_{i=1}^{m} (-1)^{i-1} \sigma_i P_{s-i,j} = \sum_{j=1}^{s-1} P_{s,j}.$$

Thus $\Phi(h_s) = h_s + \sum_{j=1}^{s} P_{s,j}$. $\square$

One of the complications of using Theorem 2 is that the product of two basis elements of $H^*(G_m(C^N); Z)$ cannot in general be readily expanded in terms of the basis. Fortunately, if we express A_i in terms of this basis, then the expressions $P_{r,j}$ in Lemma 6 will contain at most $m + 1$-fold products of the h_n's. An $m + 1$-fold

product will occur only when $k = 1$ and A_m has a term wh_1^{m-1} in its expansion. In all other cases, the monomials in $P_{r,j}$ will already be basis elements.

To avoid computation of $h_1^{m-1}h_{j-m}h_{r-j}$, Lemma 6 will be used only in the following case.

LEMMA 7. *In addition to the assumptions of Lemma 6, assume A_m has no wh_1^{m-1} term in its expansion. (This is automatic if $k > 1$.) Then if $N \geqslant 2m$ and $m \geqslant 2$,*

$$\Phi(h_{N-m+1}) \in S \text{ (the singletons) modulo } J_{k+1}$$

if and only if $A_i \in J_{k+1}$ for all i.

PROOF. By Lemma 6, $\Phi(h_{N-m+1}) \in S$ iff

$$\sum_{i=1}^{m} (-1)^{i-1} A_i \sum_{j=1}^{N-m+1} h_{j-i}h_{N-m+1-j} \quad \text{modulo } J_{k+1}$$

lies in S. Since $A_i \in J_k$, $A_i = 0$ for $i < k$. Upon expressing A_i relative to the basis of Theorem 2, noting that all products are less than or equal to m-fold, and observing $\sum_{j=1}^{N-m+1} h_{j-i}h_{N-m+1-j} = 2h_{N-m+1-i}$ + doubletons, it now follows by induction on i that $A_i \in J_{k+1}$ for all i.

PROOF OF THEOREM 1. We shall assume $N \geqslant 2m$ and $m \geqslant 2$. (The case $m = 1$ is complex projective space where the technique of this paper works almost trivially upon factoring the relation f_N^α into linear factors.)

By Theorem 3, the isomorphism f^* on equivariant cohomology is induced by an endomorphism Φ of $R[h_1, \ldots, h_m]$. Because the relations in H_T^* occur in dimensions above those of the generators, Φ is uniquely determined by f^*. Modulo J_1, Φ must give one of the automorphisms of $Z[h_1, \ldots, h_m]$ described in Lemma 5. As each is induced by an equivariant map, by composing with one of these, we may assume

$$\Phi(\sigma_i) = \sigma_i + A_i$$

where $A_i \in J_1$ for $1 \leqslant i \leqslant m$.

Suppose A_m has a term wh_1^{m-1} in its expansion where $w \in R$. Let χ be the character with weight $-w$. Then replacing α by $\alpha \otimes \chi$ replaces σ_m by the mth elementary symmetric function of $t_1 - w, t_2 - w, \ldots, t_m - w$. Modulo J_1, this is just $\sigma_m - w\sigma_{m-1}$. However it follows readily by induction that the expansion of σ_{m-1} in terms of the basis of Theorem 2 contains h_1^{m-1} with coefficient 1. Thus with the replacement of α by $\alpha \otimes \chi$, we may assume that A_m has no wh_1^{m-1} terms in it.

Now Φ must take the ideal $\langle f_{N-m+1}^\alpha, \ldots, f_N^\alpha \rangle$ into the ideal $\langle f_{N-m+1}^\beta, \ldots, f_N^\beta \rangle$. In particular, $\Phi(f_{N-m+1}^\alpha) = f_{N-m+1}^\beta$. Notice that both f_{N-m+1}^α and f_{N-m+1}^β belong to S. But modulo S and J_2, $\Phi(f_{N-m+1}^\alpha) = \Phi(h_{N-m+1})$. Hence by Lemma 7, $A_i \in J_2$. Continuing shows $A_i \in J_k$ for all k; i.e. $A_i = 0$ for all i. Thus Φ is the identity.

As Φ preserves the ideal of relations, $\Phi(f_N^\alpha) = \sum_{i=0}^{m-1} u_i f_{N-i}^\beta$. Consideration of this ideal modulo J_1 shows $u_0 = 1$ and $u_i \in J_1$ for $i > 0$. Inductively considering this relation modulo J_k for $k = 1, 2, 3, \ldots, m$ shows in fact that $u_i = 0$ for $i > 0$. Thus $\Phi(f_N^\alpha) = f_N^\beta$. But the coefficients of h_k in these relations are up to sign just the

$N - k$th symmetric functions of the weights of the two representations. Accordingly, all symmetric functions of the weights of α and β must agree. Since the weights themselves are just the roots of the polynomial $h(t) = \sum_{j=0}^{N}(-1)^j \rho_j t^{N-j}$, it is immediate that α and β have the same set of weights; i.e. that they are equivalent as linear representations.

Notice with trivial changes in dimensions that the analogous result for symplectic linear actions on quaternionic Grassmann manifolds also follows.

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