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CLOSED MAPPINGS AND QUASI-METRICS

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ABSTRACT. Closed continuous mappings with first countable images preserve quasi-metric spaces as well as nonarchimedean quasi-metric spaces and γ-spaces. This is a *strict* generalization of analogous results on perfect mappings: there exists a closed continuous mapping of a nonarchimedean quasi-metric Moore space onto a compact metric space which is neither perfect nor boundary compact.

Perfect mappings preserve quasi-metric spaces **[K2]**, **[K3]** as well as nonarchimedean quasi-metric spaces **[K1]**, **[K2]** and γ-spaces **[NC]**. It is well known that if X is a nonarchimedean quasi-metric space then X is a quasi-metric space and if X is a quasi-metric space then X is a γ-space. The first implication cannot be reversed **[K1]** and the problem of reversing the second implication is listed as Classic Problem VIII in **[TP]**. An affirmative solution of this problem would considerably simplify the treatment of mappings of quasi-metric spaces. However, only partial solutions have been obtained to the problem **[G]**, **[J1]**, **[B]**, **[F]**.

A few years ago N. Veličko announced without proof that the first countable image of a γ-space under a closed continuous mapping is also a γ-space **[V]**. It will be proved here that the first countable image of a quasi-metric space (nonarchimedean quasi-metric space, γ-space) under a closed continuous mapping is also a quasi-metric space (nonarchimedean quasi-metric space, γ-space). The most complicated case is that of quasi-metric spaces. An example will be given to show that the closed continuous image of a nonarchimedean quasi-metrizable Moore space onto a compact metric space need not be either perfect or boundary compact. Recall that the closed continuous image of a metric space is first countable if and only if the mapping is boundary compact.

1. We will use H. Junnila's neighbornet notation **[J2]**.

(1.1) Let X be a T_1-space. A binary relation $U \subset X \times X$ is an (*open*) *neighbornet* provided that $U\{x\} = \{y \in X | (x, y) \in U\}$ is an (open) neighborhood for each $x \in X$. Obviously if U is a neighborhood of the diagonal in $X \times X$, then U is a neighbornet; but the converse is false. A decreasing sequence of neighbornets $\langle U_n \rangle = \langle U_n | n \geqslant 1 \rangle$ is a *basic sequence* in X if, for each $x \in X$, the sequence $\langle U_n\{x\} \rangle$ is a neighborhood base for x. Obviously, a space X is a first countable space if and only if there is a basic sequence on X.

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(1.2) A decreasing sequence of neighbornets $\langle U_n \rangle$ is *normal* if, for each n, $U_{n+1}^2 \subset U_n$ where $U_{n+1}^2 = U_{n+1} \circ U_{n+1} = \{(x, z)|$ for some $y \in X$, $(x, y) \in U_{n+1}$ and $(y, z) \in U_{n+1}\}$. A decreasing sequence of neighbornets $\langle U_n \rangle$ is *transitive* (γ-) if, for each n, $U_n^2 = U_n$ (for each n and $x \in X$ there exists m such that $U_m^2\{x\} \subset U_n\{x\}$). H. Junnila has noted that a space is a quasi-metrizable space (nonarchimedean quasi-metrizable space, γ-space) if and only if there is a basic normal (transitive, γ-) sequence of neighbornets in X **[J2]**.

It is easy to show that if $\langle U_n \rangle$ is a basic normal (transitive, γ-) sequence on X, $x_n \in U_n\{x_n'\}$ and $\{x_m'\}$ converges to x, then $\{x_n\}$ converges to x.

(1.3) Let $\langle U_n \rangle$ be a decreasing sequence of neighbornets. Let $\hat{U}_n = \{U_{m_1} \circ U_{m_2} \circ \cdots \circ U_{m_k} | 2^{-m_1} + 2^{-m_2} + \cdots + 2^{-m_k} \leqslant 2^{-n}\}$. Let $U_n^\infty = \bigcup \{U_n^m | 1 \leqslant m < \omega_0\}$ where $U_n^m = U_n \circ U_n \circ \cdots \circ U_n$ (m times). It follows that $\langle \hat{U}_n \rangle$ is normal and $\langle U_n^\infty \rangle$ is transitive. Moreover, the sequence $\langle U_n \rangle$ is normal (transitive) if and only if $U_n = \hat{U}_n$ ($U_n = U_n^\infty$) for each n. Hence the space X is a quasi-metric space (nonarchimedean quasi-metric space, γ-space) if and only if there is a decreasing sequence of neighbornets $\langle U_n \rangle$ on X such that $\langle \hat{U}_n \rangle$ ($\langle U_n^\infty \rangle$, $\langle U_n^2 \rangle$) is basic.

2.

THEOREM. *Let f be a closed continuous mapping from a space X onto a first countable space Y. If X is a quasi-metric space (nonarchimedean quasi-metric space, γ-space), then so is Y.*

PROOF. We shall prove the case where X is a quasi-metric space using normal sequences. The other cases follow in a similar fashion using transitive sequences and γ-sequences.

Let $\langle U_n \rangle$ be a basic normal sequence on X (1.2) and let $\langle V_n \rangle$ be a basic sequence on Y (1.1). Define *open* neighbornets $\tilde{U}_n \subset U_n$ such that $f(\tilde{U}_n(f^{-1}(y))) \subset V_n\{y\}$ for each $y \in Y$ by letting

$$\tilde{U}_n\{x\} = \operatorname{Int}\big(U_n\{x\} \cap f^{-1}(V_n\{f(x)\})\big).$$

Define another basic sequence $\langle W_n \rangle$ in Y by letting

$$\begin{aligned} W_n\{y\} &= \{y' \in Y | f^{-1}(y') \subset \tilde{U}_n(f^{-1}(y))\} \\ &= Y - f(X - \tilde{U}_n(f^{-1}(y))) \subset f(\tilde{U}_n(f^{-1}(y))) \subset V_n\{y\}. \end{aligned}$$

Since f is a closed map, it is clear that $W_n\{y\}$ is an open set and, since $W_n\{y\} \subset V_n\{y\}$, it is also clear that $\langle W_n \rangle$ is a basic sequence.

It follows from (1.3) that Y is quasi-metrizable if $\langle \hat{W}_n \rangle$ is basic.

Suppose $\langle \hat{W}_n \rangle$ is not basic. Then there exists a sequence $\langle y_n \rangle$ with $y_n \in \hat{W}_n\{y\}$ for each n such that y is not a limit point of $\langle y_n \rangle$.

By (1.3) for each n there exists a finite sequence of points $y = y_n^0, y_n^1, \ldots, y_n^k = y_n$ and integers $m_1, \ldots, m_k$ (where k and each m_i depend on n) such that $2^{-m_1} + 2^{-m_2} + \cdots + 2^{-m_k} \leqslant 2^{-n}$ and $y_n^i \in w_{m_i}\{y_n^{i-1}\}$ for $0 < i \leqslant k$. Notice we can always choose $y_n^1 \neq y_n^0$. Pick $x_n \in f^{-1}(y_n)$ and choose a finite sequence $x_n = x_n^k, x_n^{k-1}, \ldots, x_n^1$ such that $x_n^i \in f^{-1}(y_n^i)$ and $x_n^i \in \tilde{U}_{m_i}\{x_n^{i-1}\}$ for $0 < i \leqslant k$. To do this suppose we have found $x_n^k, \ldots, x_n^i$. Since $y_n^i \in W_{mi}\{y_n^{i-1}\}$, we have $x_n^i \in f^{-1}(y_n^i) \subset \tilde{U}_{m_i}\{f^{-1}(y_n^{i-1})\}$. Choose $x_n^{i-1} \in f^{-1}(y_n^{i-1})$ such that $x_n^i \in \tilde{U}_{m_i}\{x_n^{i-1}\}$. We have $x_n = x_n^k \in \tilde{U}_{m_k} \circ \cdots \circ \tilde{U}_{m_2}\{x_n^1\} \subset U_{m_k} \circ \cdots \circ U_{m_2}\{x_n^1\} \subset \hat{U}_n\{x_n^1\}$. Thus $x_n = x_n^k \in \hat{U}_n\{x_n^1\}$. Since $\langle U_n \rangle$ is normal we have $\hat{U}_n = U_n$ by (1.3), and $x_n \in U_n\{x_n^1\}$. We also have

$$y_n^1 \in W_{m_1}\{y\} \subset W_n\{y\}$$

and, since $\langle W_n \rangle$ is basic, the sequence $\langle y_n^1 \rangle$ converges to y. Since $x_n^1 \in f^{-1}(y_n^1)$, $y_n^1 \neq y$ and f is a closed map, there exists a limit point x of $\langle x_n^1 \rangle$ and $x \in f^{-1}(y)$. Otherwise there exists some open set $G \supset f^{-1}(y)$ and G does not intersect $\{x_1^1, x_2^1, \ldots\}$ and $Y - f(X - G)$ does not intersect $\{y_1^1, y_2^1, \ldots\}$. Since x is a limit point of $\langle x_n^1 \rangle$, $x_n \in U_n\{x_n^1\}$ and $\langle U_n \rangle$ is basic and normal, it follows by (1.2) that x is a limit point of $\langle x_n \rangle$. Thus $y = f(x)$ is a limit point of $\langle y_n \rangle = \langle f(x_n) \rangle$. From this contradiction we see that $\langle \hat{W}_n \rangle$ is basic and Y is a quasi-metric space.

3. The following example was obtained jointly by R. W. Heath and the author.

Example. There is a closed mapping of a nonarchimedean quasi-metrizable Moore space onto a compact metric space which fails to be either perfect or boundary compact.

The space Ψ **[GJ]** is the domain space. The underlying set of Ψ is a countable set N and an infinite maximal collection of infinite subsets of N such that the intersection of any two subsets is finite. The points of N are isolated while a basic neighborhood of any x in $\Psi - N$ is x and all but finitely many elements of x. (Recall that if $x \in \Psi - N$, then x is an infinite subset of N.) If we define $U_n\{x\} = \{x\}$ if $x \in N$, and if $x = \{x_1, x_2, \ldots\} \in X - N$, let $U_n\{x\} = \{x\} \cup \{x_n, x_{n+1}, \ldots\}$. Then the sequence $\langle U_n \rangle$ is basic and transitive. If $G_n = \{U_n\{x\} | x \in X\}$ then $\{G_n | 1 \leq n < \omega_0\}$ is a development for Ψ.

The range space is $\Psi/\Psi - N$. This space is a convergent sequence. Notice that $\Psi - N$ has no interior points and is not compact. The obvious quotient map is closed.

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