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FINITE ASPHERICAL COMPLEXES WITH INFINITELY-GENERATED GROUPS OF SELF-HOMOTOPY-EQUIVALENCES

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ABSTRACT. A finite aspherical complex X is constructed whose group of homotopy classes of basepoint-preserving self-homotopy-equivalences is infinitely generated.

0. Introduction. Let EX denote the H-space of basepoint-preserving homotopy equivalences from X to X. The set $\pi_0(EX)$ becomes a group with multiplication induced by composition of representatives; it is denoted $\mathcal{E}(X)$ and called, by abuse of language, the *group of homotopy equivalences* of X.

It was proved independently by D. Sullivan **[Su]** and C. Wilkerson **[W]** that $\mathcal{E}(X)$ is finitely presented when X is a simply-connected finite complex. In contrast, Frank and Kahn **[F-K]** showed that $\mathcal{E}(S^1 \vee S^p \vee S^{2p-1})$ is infinitely generated for $p \geqslant 2$. They also suggested the following alternative method for producing examples. Start with Lewin's **[L]** example of a finitely-presented group G with Aut(G) infinitely generated. It is well known (see e.g. **[S**, p. 427**]**) that if X is a $K(\pi, 1)$-complex then $\mathcal{E}(X) \cong \mathrm{Aut}(\pi)$, and therefore any $K(G, 1)$-complex would have an infinitely-generated group of homotopy equivalences. In this note, I will construct a *finite* $K(G, 1)$-complex X, note some of its salient features, and raise some questions.

1. Construction of X. Let G_Y be the group with presentation

$$P_Y = \langle a, b, x \colon [a, b] = 1, x^{-1}ax = a^2, x^{-1}bx = b^2 \rangle$$

and let G_M be the group with presentation

$$P_M = \langle a, b, y \colon [a, b] = 1, y^{-1}ay = a, y^{-1}by = ba \rangle.$$

We will construct a $K(G_Y, 1)$-complex Y and a $K(G_M, 1)$-complex M.

Let exp: $\mathbf{C} \to \mathbf{C}$ be the exponential map. Let $T = S^1 \times S^1 \subset \mathbf{C}^2$ and let p_a, p_b: $I \to T$ be the loops $p_a(t) = (\exp(2\pi i t), 1)$ and $p_b(t) = (1, \exp(2\pi i t))$. Then $a = \langle p_a \rangle$ and $b = \langle p_b \rangle$ generate $\pi_1(T, (1, 1)) \cong \mathbf{Z} \oplus \mathbf{Z}$. If f: $T \to T$ is the 4-fold covering map defined by

$$f(\exp(2\pi i t_1), \exp(2\pi i t_2)) = (\exp(4\pi i t_1), \exp(4\pi i t_2))$$

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then $f_\#(a) = a^2$ and $f_\#(b) = b^2$. Let Y be the mapping torus

$$Y = T \times I / (w, 0) \sim (f(w), 1).$$

Let $p_x\colon I \to Y$ be the loop $p_x(t) = [(1, 1), t]$, and let $x = \langle p_x \rangle \in \pi_1(Y, [(1, 1), 0])$. The fundamental group of Y is a Higman-Neumann-Neumann (HNN) group with presentation P_Y. To see directly that Y is aspherical, consider its infinite cyclic covering space $\tilde{Y}$ corresponding to the commutator subgroup (= the normal closure of a and b) of G_Y. It consists of countably many 3-dimensional aspherical pieces attached along aspherical subsets, and we use Theorem 5 of **[Wh]** and direct limits to see that $\tilde{Y}$ is aspherical. Therefore Y is a 3-dimensional finite $K(G_Y, 1)$-complex.

Let $h\colon T \to T$ be the homeomorphism given by

$$h(\exp(2\pi i t_1), \exp(2\pi i t_2)) = (\exp(2\pi i(t_1 + t_2)), \exp(2\pi i t_2)).$$

Then $h_\#(a) = a$ and $h_\#(b) = ba$. Form the closed aspherical 3-manifold

$$M = T \times I / (w, 0) \sim (h(w), 1).$$

It is a bundle over the circle with torus fiber. Let $y \in \pi_1(M, [(1, 1), 0])$ be $\langle p_y \rangle$ where $p_y(t) = [(1, 1), t]$. Then $\pi_1(M)$ is an HNN group with presentation P_M, so M is a 3-dimensional $K(G_M, 1)$-manifold.

The complex X can be writen as a mapping torus in two ways.

$$X = Y \times I / \big[[w, r], 0\big] \sim \big[[h(w), r], 1\big]$$

or

$$X = M \times I / \big[[v, s], 0\big] \sim \big[[f(v), s], 1\big].$$

X is 4-dimensional and is aspherical by the same reasoning as we used for Y. The fundamental group of X is an HNN group of G with presentation

$$P_G = \langle a, b, x, y\colon [a, b] = 1, [x, y] = 1, x^{-1}ax = a^2, \\ x^{-1}bx = b^2, y^{-1}ay = a, y^{-1}by = ba \rangle$$

so X is a finite $K(G, 1)$-complex.

2. Properties of X. (1) It was shown by Lewin **[L]** that $\mathrm{Aut}(G)$ is infinitely generated. Therefore $\mathcal{E}(X) \cong \mathrm{Aut}(G)$ is infinitely generated.

(2) By 9.16(b) of **[B]**, G is a duality group of dimension 4 over $\mathbf{Z}$, therefore X is a duality complex. But G is not a Poincaré duality group, since $H^4(G; \mathbf{Z}G) \cong H^4_{\text{finite}}(\tilde{X}; \mathbf{Z})$ is an infinitely-generated group, so X is not a Poincaré complex.

(3) By making a few small changes we can produce infinitely many similar examples. There are groups G_m for all integers $m \geq 1$, with $G_1 = G$, such that

(a) $\mathrm{Aut}(G_m)$ is infinitely generated.

(b) $G_m \cong G_n$ if and only if $m = n \cdot 2^k$ for some $k \in \mathbf{Z}$.

Presentations for the G_m and a proof of (b) are given in an addendum to this note. The proof of (a) is a slight modification of the argument of Lewin **[L]**. For each $m \geq 1$, by replacing the map $h = h_1$ of §1 by $h_m\colon T \to T$ where

$$h_m(\exp(2\pi i t_1), \exp(2\pi i t_2)) = (\exp(2\pi i(t_1 + m \cdot t_2)), \exp(2\pi i t_2)),$$

we produce a closed aspherical 3-manifold M_m contained in a 4-dimensional aspherical complex X_m with $\pi_1(X_m) \cong G_m$. By (b), infinitely many homotopy types appear among the X_m. The manifolds M_m appear in a different context in [**R-S**].

3. Questions. (1) The Frank and Kahn examples and the X_m are dissimilar examples of finite complexes Z with $\mathfrak{E}(Z)$ infinitely generated. Are there other types of examples? Is there a 2-dimensional example?

(2) Are there examples for which infinitely many generators of $\mathfrak{E}(Z)$ can be represented by homeomorphisms? (There are closed 3-manifolds M for which $\pi_1(EM)$ is infinitely generated and infinitely many generators can be represented by isotopies [**M**].)

(3) Can a closed $K(\pi, 1)$-manifold N have $\mathfrak{E}(N)$ infinitely generated?

(4) The groups G_m are duality groups over $\mathbf{Z}$ but not Poincaré duality groups. Can a Poincaré duality group have an infinitely-generated automorphism group? An affirmative answer to (3) would provide an example of such a group [**B**, p. 170]. It is conceivable that (3) and (4) are equivalent.

(5) If π is a finitely-presented group with $\mathrm{Aut}(\pi)$ infinitely generated, must π contain infinitely-divisible elements?

Addendum: The groups G_m. For $m \geqslant 1$ let G_m be the group with presentation

$$\langle a, b, x, y \colon [a, b] = 1, [x, y] = 1, x^{-1}ax = a^2, x^{-1}bx = b^2, y^{-1}ay = a, y^{-1}by = ba^m\rangle.$$

Let $\mathbf{Z}[\frac{1}{2}]$ be the additive group of dyadic rationals. We regard G_m as a semidirect product

$$1 \to \mathbf{Z}\left[\tfrac{1}{2}\right] \oplus \mathbf{Z}\left[\tfrac{1}{2}\right] \to G_m \to \mathbf{Z} \oplus \mathbf{Z} \to 1$$

where $\mathbf{Z} \oplus \mathbf{Z}$ is the abelianization of G_m, generated by x and y, and $\mathbf{Z}[\frac{1}{2}] \oplus \mathbf{Z}[\frac{1}{2}]$ is the commutator subgroup where a and b correspond to $(1, 0)$ and $(0, 1)$ respectively. The choice of basis $\{a, b\}$ determines an isomorphism $\mathrm{Aut}(\mathbf{Z}[\frac{1}{2}] \oplus \mathbf{Z}[\frac{1}{2}]) \cong GL_2(\mathbf{Z}[\frac{1}{2}])$ so that the action of $\mathbf{Z} \oplus \mathbf{Z}$ is given by $\phi_m \colon \mathbf{Z} \oplus \mathbf{Z} \to GL_2(\mathbf{Z}[\frac{1}{2}])$ defined by

$$\phi_m(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \phi_m(y) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

PROPOSITION. *$G_m \cong G_n$ if and only if $m = n \cdot 2^i$ for some integer i.*

PROOF. Our proof uses elements from [**L**].

Suppose $f \colon G_m \cong G_n$. Since $\mathbf{Z}[\frac{1}{2}] \oplus \mathbf{Z}[\frac{1}{2}]$ is the commutator subgroup we have a morphism of extensions

$$\begin{array}{ccccccccc} 1 \to \mathbf{Z}\left[\frac{1}{2}\right] & \oplus & \mathbf{Z}\left[\frac{1}{2}\right] & \to & G_m & \to & \mathbf{Z} & \oplus & \mathbf{Z} \to 1 \\ & \downarrow f_1 & & & \downarrow f & & & \downarrow f_2 & \\ 1 \to \mathbf{Z}\left[\frac{1}{2}\right] & \oplus & \mathbf{Z}\left[\frac{1}{2}\right] & \to & G_n & \to & \mathbf{Z} & \oplus & \mathbf{Z} \to 1 \end{array}$$

We choose generators A, B, X, Y for G_n so that

$$\phi_n(X) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \phi_n(Y) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

We regard f_1 as an element of $GL_2(\mathbf{Z}[\frac{1}{2}])$, written with respect to the bases $\{a, b\} \subset \mathbf{Z}[\frac{1}{2}] \oplus \mathbf{Z}[\frac{1}{2}] \subset G_m$ and $\{A, B\} \subset \mathbf{Z}[\frac{1}{2}] \oplus \mathbf{Z}[\frac{1}{2}] \subset G_n$. Commutativity of the diagram implies that

$$f_1 \phi_m(x) f_1^{-1} = \phi_n(f_2(x)),$$

therefore $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) = \phi_n(f_2(x))$ so $f_2(x) = X$. Since f_2 is an isomorphism, $f_2(y) = X^k Y^{\pm 1}$. Therefore

$$f_1 \phi_m(y) f_1^{-1} = \phi_n(X^k Y^{\pm 1}) = \begin{pmatrix} 2^k & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & \pm n \\ 0 & 1 \end{pmatrix}.$$

Since the determinant of $\phi_m(y)$ is 1, we must have $k = 0$, so

$$f_1 \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} f_1^{-1} = \begin{pmatrix} 1 & \pm n \\ 0 & 1 \end{pmatrix}.$$

Writing $f_1 = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, this equation becomes

$$\frac{1}{ps - rq} \begin{pmatrix} ps - rq - mpr & mp^2 \\ -mr^2 & ps - qr + mpr \end{pmatrix} = \begin{pmatrix} 1 & \pm n \\ 0 & 1 \end{pmatrix}.$$

Therefore $r = 0$ and we have

$$\begin{pmatrix} ps & mp^2 \\ 0 & ps \end{pmatrix} = \begin{pmatrix} ps & \pm nps \\ 0 & ps \end{pmatrix}$$

so $mp^2 = \pm nps$. Since ps was a unit in the ring $\mathbf{Z}[\frac{1}{2}]$, p and s are powers of 2 and we have $m = n \cdot 2^i$ for some i.

Conversely, if $m = n \cdot 2^i$ we define a homomorphism $f\colon G_m \to G_n$ on generators by $f(a) = X^i A X^{-i}$, $f(b) = B$, $f(x) = X$, $f(y) = Y$. The inverse of f is g defined by $g(A) = x^{-i} a x^i$, $g(B) = b$, $g(X) = x$, $g(Y) = y$, so $G_m \cong G_n$. □

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