PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, October 1980

ON S-CLOSED SPACES

JAMES E. JOSEPH AND MYUNG H. KWACK

ABSTRACT. In this paper, we initially give several new characterizations of the class of S-closed spaces, which was introduced by T. Thompson [Proc. Amer. Math. Soc. **60** (1976), 335–338]. We then employ these characterizations to produce analogues for S-closed spaces of the well-known theorem from real analysis that an upper-semicontinuous real-valued function on a closed interval assumes a maximum, and of two well-known theorems of G. Birkhoff and A. D. Wallace, which established that each upper-semicontinuous function from a compact space into a partially ordered set assumes a maximal value and that each compact space has a maximal element with respect to each upper-semicontinuous quasi order on the set. The statements in these latter analogues are then shown to characterize S-closed spaces. A "fixed set theorem" for multifunctions on S-closed spaces is also established.

Introduction. In [$\mathbf{T_1}$], Thompson introduced the class of S-closed spaces. The class of S-closed spaces is easily seen to be a subclass of the class of $H(i)$ spaces of Scarborough and Stone [**S-S**], and Thompson proved in [$\mathbf{T_1}$] that for compact Hausdorff spaces, the concept of S-closed is equivalent to the concepts of extremally disconnectedness and projectiveness. S-closed spaces were defined by Thompson in terms of the notion of semiopen sets, which was initiated by Levine [**L**] and studied extensively by Crossley and Hildebrand [$\mathbf{C\text{-}H_1}$], [$\mathbf{C\text{-}H_2}$]. In [$\mathbf{T_2}$], Thompson produced an analogue for S-closed spaces of the well-known definition of Alexandroff and Urysohn [**A-U**] that a Hausdorff space is H-closed if it is a closed subspace of any Hausdorff space which contains it. Recently, Herrmann [$\mathbf{H_1}$] has constructed an explicit example of a noncompact Hausdorff S-closed space, and has treated S-closed spaces from a nonstandard viewpoint in [$\mathbf{H_2}$]. Cameron [**C**] has shown that S-closed spaces may be characterized as spaces where covers by regular-closed sets have finite subcovers.

In §1 of this paper we offer several new characterizations of S-closed spaces. These characterizations are given in terms of nets with well-ordered directed sets, in terms of a generalization of complete accumulation point and in terms of graphs of functions into the space. These characterizations parallel ones which have been found to be useful for compact spaces and some which have been given recently for other compactness generalizations such as H-closed, minimal Hausdorff, Urysohn-closed, and minimal Urysohn in [$\mathbf{H\text{-}L_1}$], [$\mathbf{H\text{-}L_2}$], [$\mathbf{J_1}$]–[$\mathbf{J_3}$].

In §2, results of §1 are employed to give analogues for S-closed spaces of two well-known theorems of Birkhoff [**B**] and Wallace [**W**] for compact spaces. Frank-

Received by the editors May 15, 1978 and, in revised form, May 1, 1979.
AMS (*MOS*) *subject classifications* (1970). Primary 54D20, 54D25, 54D30; Secondary 54F05.
Key words and phrases. S-closed, regular-closed, semiopen, quasi order, partial order, upper- (lower-) semicontinuity, maximal and minimal elements.

© 1980 American Mathematical Society
0002-9939/80/0000-0529/$03.00

lin [F] has pointed out that a result of Ceder [Ce] yields converses to the theorems of Birkhoff and Wallace and, consequently, provides two characterizations of compactness. Our analogues for S-closed spaces are shown to characterize S-closed spaces. We also utilize the results of §1 to produce a "fixed set theorem" for S-closed spaces.

Preliminaries. Let X be a space, let $A \subset X$, let $x \in X$ and let Ω be a filterbase on X; $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\Sigma(A)$, and $\mathrm{ad}\,\Omega$ will represent, respectively, the closure of A, interior of A, family of open subsets which contain A, and adherence of Ω. A is *semiopen* if some open subset V of X satisfies $V \subset A \subset \mathrm{cl}(V)$. It is readily seen that A is semiopen if and only if $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A))$. It is also easy to see that $\mathrm{cl}(A)$ is semiopen when A is semiopen. A is regular-closed if $A = \mathrm{cl}(\mathrm{int}(A))$ and is regular-open if $X - A$ is regular-closed. The family of regular-closed (semiopen) subsets of X about x is denoted by $\mathrm{RC}(x)$ ($\mathrm{SO}(x)$); x is in the ***θ-semiclosure of A*** ($x \in \theta\text{-}\mathrm{cl}_s(A)$) if each $V \in \mathrm{SO}(x)$ satisfies $\mathrm{cl}(V) \cap A \neq \varnothing$. A is ***θ-semiclosed*** if $A = \theta\text{-}\mathrm{cl}_s(A)$. It follows easily that $\theta\text{-}\mathrm{cl}_s(P) \subset \theta\text{-}\mathrm{cl}_s(Q)$ when $P \subset Q \subset X$ and, since $\mathrm{cl}(A) \in \mathrm{SO}(x)$ when $A \in \mathrm{SO}(x)$, that $\theta\text{-}\mathrm{cl}_s(P)$ is θ-semiclosed for each $P \subset X$. A regular-closed subset is semiopen and if A is semiopen the equation $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))) = \mathrm{cl}(\mathrm{int}(A)) = \mathrm{cl}(A)$ holds. Consequently $\mathrm{RC}(x) = \{\mathrm{cl}(V)\colon V \in \mathrm{SO}(x)\}$. Therefore, $x \in \theta\text{-}\mathrm{cl}_s(A)$ if and only if each $R \in \mathrm{RC}(x)$ satisfies $A \cap R \neq \varnothing$. In view of the preceding remarks, the definitions of Thompson [T_1] may be rephrased as follows. Ω *s-accumulates to* x ($x \in \theta\text{-}\mathrm{ad}_s\,\Omega$) if and only if $x \in \theta\text{-}\mathrm{cl}_s(F)$ for each $F \in \Omega$ if and only if each $F \in \Omega$ and $R \in \mathrm{RC}(x)$ satisfy $R \cap F \neq \varnothing$. Ω *s-converges to* x if and only if for each $R \in \mathrm{RC}(x)$ there is an $F \in \Omega$ satisfying $F \subset R$. The corresponding statements using nets are obvious.

It is well known that the collection of regular-open subsets of (X, T) is a base for a topology, T_s, on X. In our first proposition we establish a connection between θ-semiclosed subsets and T_s. If ∇ is a family of subsets of X we will denote the intersection of the sets in ∇ by $\bigcap \nabla$. No proofs are given in this section.

PROPOSITION 1. *A subset K of a space (X, T) is θ-semiclosed if and only if $K = \bigcap \nabla(K)$, where $\nabla(K) = \{V \in T_s\colon K \subset V\}$.*

COROLLARY 1. *A regular open subset of a space is θ-semiclosed.*

If X and Y are spaces, a function $\lambda\colon X \to Y$ is *(θ, s)-continuous at* $x \in X$ if for each $W \in \mathrm{SO}(\lambda(x))$ there is a $V \in \Sigma(x)$ with $\lambda(V) \subset \mathrm{cl}(W)$. λ is *(θ, s)-continuous on X* (or simply "*(θ, s)-continuous*") if λ is (θ, s)-continuous at each $x \in X$.

THEOREM 1. *The following statements are equivalent for a function λ from a space X to a space Y.*

(a) *The function λ is (θ, s)-continuous.*

(b) *Each filterbase Ω on X satisfies* $\lambda(\mathrm{ad}\,\Omega) \subset \theta\text{-}\mathrm{ad}_s\,\lambda(\Omega)$.

(c) *Each $A \subset X$ satisfies* $\lambda(\mathrm{cl}(A)) \subset \theta\text{-}\mathrm{cl}_s(\lambda(A))$.

(d) *Each $A \subset Y$ satisfies* $\mathrm{cl}(\lambda^{-1}(A)) \subset \lambda^{-1}(\theta\text{-}\mathrm{cl}_s(A))$.

(e) *For each θ-semiclosed $A \subset Y$, $\lambda^{-1}(A)$ is closed in X.*

(f) *For each $R \in \mathrm{RC}(\lambda(x))$ there is a $V \in \Sigma(x)$ with $\lambda(V) \subset R$.*

1. Characterizations of S-closed spaces. A function $\lambda: X \to Y$ has a *θ-semisubclosed graph* if $\theta\text{-ad}_s\, \lambda(\Omega) \subset \{\lambda(x)\}$ for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$. This definition is motivated by the known result that λ has a closed graph if and only if (1) $\{\lambda(x)\}$ is closed in Y and (2) $\text{ad}\, \lambda(\Omega) \subset \{\lambda(x)\}$ for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$. If X is a nonempty set, $x_0 \in X$ and Ω is a filterbase on X, then X with the topology, $\{A \subset X: x_0 \in X - A$ or $F \subset A$ for some $F \in \Omega\}$, is denoted by $X(x_0, \Omega)$. It is noted in **[J₁]** that $X(x_0, \Omega)$ is Hausdorff, completely normal, and fully normal if Ω has empty intersection on $X - \{x_0\}$. A point x in a space X is an *s-complete accumulation point* of $A \subset X$ if $|A \cap R| = |A|$ for each $R \in \text{RC}(x)$, where $|A|$ denotes the cardinality of the set A. If $\psi: X \to Z$, $\lambda: X \to Z$ and $\alpha: Y \to Z$ are functions, we shall use the following notations: $G(\psi) = \{(x, \psi(x)): x \in X\}$, $\mathfrak{S}(\psi, \alpha, X \times Y, Z) = \{(x, y) \in X \times Y: \psi(x) = \alpha(y)\}$, $E(\psi, \lambda, X, Z) = \{x \in X: \psi(x) = \lambda(x)\}$.

The following definition is used as a primitive.

Definition **[T₁]**. A space is *S-closed* if each net on the space s-accumulates to some point in the space.

For a filterbase Ω on a space, $\theta\text{-ad}_s\, \Omega$ will be called the *θ-semiadherence of* Ω. Ordinals and cardinals are defined, for example, as in Kelley **[K]**.

Theorem 2. *The following statements are equivalent for a space Z.*

(a) *Z is S-closed.*

(b) *Each net on Z with a well-ordered directed set s-accumulates to some point in Z.*

(c) *Each infinite subset of Z has an s-complete accumulation point.*

(d) *Each family of θ-semiclosed subsets of Z with the finite intersection property has a nonempty intersection.*

(e) *Each filterbase on Z has a nonempty θ-semiadherence* **[T₁]**.

(f) *Each filterbase on Z with at most one θ-semiadherent point is s-convergent.*

(g) *For all spaces X, all functions $\psi: X \to Z$ with θ-semisubclosed graphs are (θ, s)-continuous.*

(h) *For all spaces X, Y and all functions $\psi: X \to Z$ and $\alpha: Y \to Z$ with θ-semisubclosed graphs, $\mathfrak{S}(\psi, \alpha, X \times Y, Z)$ is closed in $X \times Y$.*

(i) *Same as* (h), *except that $X = Y$.*

(j) *Same as* (i), *except that the conclusion is that $E(\psi, \alpha, X, Z)$ is closed in X.*

(k) *Same as* (i), *except that the conclusion is that $E(\psi, \alpha, X, Z) = X$ whenever $E(\psi, \alpha, X, Z)$ is dense in X.*

Proof. The proofs that (a) implies (b), that (h) implies (i) and that (j) implies (k) are obvious.

Proof that (b) *implies* (c). An infinite subset A of Z may be assumed to be a net with a well-ordered index set. Let p be an s-accumulation point of A; p is clearly an s-complete accumulation point of A.

Proof that (c) *implies* (d). If (d) does not hold, let μ be the smallest cardinal for which some family of θ-semiclosed subsets, $\Omega = \{F(\alpha): \alpha \in \mu\}$, with the finite intersection property has $\bigcap \Omega = \varnothing$. For each $\alpha \in \mu$ let $\Omega(\alpha) = \{F(\beta): \beta < \alpha\}$. If

$\beta < \alpha < \mu$ then $\bigcap \Omega(\alpha) \subset \bigcap \Omega(\beta)$. We first show that $|\bigcap \Omega(\alpha)| \geq \mu$ for each $\alpha < \mu$. Suppose that $|\bigcap \Omega(\alpha)| < \mu$ for some $\alpha < \mu$. For each $x \in \bigcap \Omega(\alpha)$ choose an $F(\alpha(x)) \in \Omega$ with $x \notin F(\alpha(x))$ and let $\Omega^* = \Omega(\alpha) \cup \{F(\alpha(x)): x \in \bigcap \Omega(\alpha)\}$. Then $|\Omega^*| < \mu$ and $\bigcap \Omega^* = \varnothing$. This is a contradiction. Hence it is possible to choose, for each $\alpha \in \mu$, $x(\alpha) \in \bigcap \Omega(\alpha)$ with $x(\alpha) \neq x(\beta)$ for $\beta \in \alpha$. Let x be an s-complete accumulation point of $A = \{x(\alpha)\}$. Let $R \in \mathrm{RC}(x)$ and $\alpha \in \mu$. Then $|R \cap A| = |A| = \mu$. There is a β with $\alpha \in \beta \in \mu$ such that $x(\beta) \in R \cap \bigcap \Omega(\beta) \subset R \cap F(\alpha)$. So $x \in \theta\text{-cl}_s(F(\alpha)) = F(\alpha)$. Thus $x \in \bigcap \Omega$, a contradiction.

Proof that (d) *implies* (e). Let Ω be a filterbase on Z. Then $\Omega^* = \{\theta\text{-cl}_s(F): F \in \Omega\}$ is a family of θ-semiclosed subsets of Z with the finite intersection property. Hence $\theta\text{-ad}_s\, \Omega = \bigcap \Omega^* \neq \varnothing$.

Proof that (e) *implies* (f). Let Ω be a filterbase on Z and let $x \in Z$ with $\theta\text{-ad}_s\, \Omega \subset \{x\}$. Then $\theta\text{-ad}_s\, \Omega = \{x\}$ from (e). Let $R \in \mathrm{RC}(x)$ and suppose for all $F \in \Omega$, $F \cap (Z - R) \neq \varnothing$. Then $\Omega^* = \{F - R: F \in \Omega\}$ is a filterbase on Z, so $\varnothing \neq \theta\text{-ad}_s\, \Omega^* \subset \{x\} - R$. This is a contradiction, so there is an $F \in \Omega$ with $F \subset R$.

Proof that (f) *implies* (g). Let Ω be a filterbase on X and $y \in \psi(\mathrm{ad}\, \Omega)$. Let $x \in \mathrm{ad}\, \Omega$ with $y = \psi(x)$. If $\Omega^* = \{(V \cap F) - \{x\}: V \in \Sigma(x) \text{ and } F \in \Omega\}$ is not a filterbase on $X - \{x\}$, then $x \in F$ for each $F \in \Omega$. So $y = \psi(x) \in \psi(F)$ and $y \in \theta\text{-ad}_s\, \psi(\Omega)$. If Ω^* is a filterbase on $X - \{x\}$, then $\Omega^* \to x$ and $\theta\text{-ad}_s\, \psi(\Omega^*) \subset \{\psi(x)\}$. Therefore, by (f), $y \in \theta\text{-ad}_s\, \psi(\Omega^*) \subset \theta\text{-ad}_s\, \psi(\Omega)$.

Proof that (g) *implies* (h). Let (p, q) be a limit point of $\mathfrak{S}(\psi, \alpha, X \times Y, Z)$ and Ω be a filterbase on $\mathfrak{S}(\psi, \alpha, X \times Y, Z) - \{(p, q)\}$ with $\Omega \to (p, q)$. If π_X and π_Y represent the projections of $X \times Y$ onto X and Y, respectively, it is easily seen that $\psi(\pi_X(F)) = \alpha(\pi_Y(F))$ for each $F \in \Omega$. First suppose $\pi_X(F) = \{p\}$ for some $F \in \Omega$. Then we may assume without loss that $\pi_Y(\Omega)$ is a filterbase on $Y - \{q\}$. Since $\pi_Y(\Omega) \to q$ we have $\theta\text{-ad}_s\, \alpha(\pi_y(\Omega)) \subset \{\alpha(q)\}$. We see easily that $\{\psi(p)\} \subset \theta\text{-ad}_s\, \psi(\pi_X(\Omega)) = \theta\text{-ad}_s\, \alpha(\pi_Y(\Omega))$. Hence $\psi(p) = \alpha(p)$ and $(p, q) \in \mathfrak{S}(\psi, \alpha, X \times Y, Z)$. A similar argument applies if $\pi_Y(F) = \{q\}$ for some $F \in \Omega$. In the remaining case, we assume without loss that $\pi_X(\Omega)$ and $\pi_Y(\Omega)$ are filterbases on $X - \{p\}$ and $Y - \{q\}$, respectively. Since $\pi_X(\Omega) \to p$ and $\pi_Y(\Omega) \to q$, and ψ, α have θ-semisubclosed graphs, we have from (g) that $\theta\text{-ad}_s\, \alpha(\pi_Y(\Omega)) = \{\alpha(q)\}$. Since $\theta\text{-ad}_s\, \alpha(\pi_Y(\Omega)) = \theta\text{-ad}_s\, \psi(\pi_X(\Omega)) \subset \{\psi(p)\}$ we get $\psi(p) = \alpha(q)$ and $(p, q) \in \mathfrak{S}(\psi, \alpha, X \times Y, Z)$.

Proof that (i) *implies* (j). Let Δ be the diagonal in $X \times X$. Then π_X restricted to Δ is a homeomorphism and $\mathfrak{S}(\psi, \alpha, X \times X, Z) \cap \Delta$ is closed in Δ. $E(\psi, \alpha, X, Z) = \pi_X(\mathfrak{S}(\psi, \alpha, X \times X, Z) \cap \Delta)$, which is closed in X.

Proof that (k) *implies* (a). Suppose Z is not S-closed and let η be a net in Z with no s-accumulation point and let Ω be the filterbase generated by η. Then $\theta\text{-ad}_s\, \Omega = \varnothing$. Choose $x_0, z_0 \in Z$ with $x_0 \neq z_0$. Define $\psi, \alpha: Z(z_0, \Omega) \to Z$ by $\psi(x) = x$ for all $x \in Z$ and $\alpha(x) = x$ for $x \neq z_0$ and $\alpha(z_0) = x_0$. Then $E(\psi, \alpha, Z(z_0, \Omega), Z) = Z - \{z_0\}$ which is dense in $Z(z_0, \Omega)$. We will show that Z does not satisfy (k) by showing that ψ and α have θ-semisubclosed graphs. Let $p \in Z$ and Ω^* be a filterbase on $Z - \{p\}$ such that $\Omega^* \to p$ in $Z(z_0, \Omega)$. Then $p = z_0$. So Ω^* is a filterbase on $Z - \{z_0\}$ and $\theta\text{-ad}_s\, \psi(\Omega^*) = \theta\text{-ad}_s\, \Omega^* \subset \theta\text{-ad}_s\, \Omega = \varnothing \subset \{\psi(x)\}$. This shows ψ has a θ-semisubclosed graph. Similarly $\theta\text{-ad}_s\, \alpha(\Omega^*) = \theta\text{-ad}_s\, \Omega^* \subset \theta\text{-ad}_s\, \Omega = \varnothing \subset \{\alpha(x)\}$.

This completes the proof of Theorem 2.

REMARK 1. If replacements of phrases are made in Theorem 2 as indicated below, the statement resulting from such replacements is valid (see **[J₁]**, **[J₂]**). Let $\mathfrak{T}$ be any class of spaces containing as a subclass the Hausdorff, completely normal, fully normal spaces.

(i) *In* (g), (h), (i), *or* (j), *replace "function*(*s*)*" by "bijection*(*s*)*".*

(ii) *In* (k), *replace "functions" by "functions* (*one a bijection*)*".*

(iii) *In* (g), (h), (i), (j), *or* (k), *replace the requirement "all spaces" by "all spaces in class* $\mathfrak{T}$*".*

Our next characterization of S-closed spaces is established separately as it does not appear to fit easily into the scheme of proof used in Theorem 2.

THEOREM 3. *A necessary and sufficient condition for a space Z to be S-closed is that for all spaces X, $\mathcal{E}(\psi, \psi, X \times X, Z)$ is a closed subset of $X \times X$ for all functions ψ: $X \to Z$ with θ-semisubclosed graphs.*

PROOF. The proof of necessity is obvious from (i) of Theorem 2. For the sufficiency, suppose Ω is a filterbase on Z with $\theta\text{-ad}_s\, \Omega = \varnothing$. We may choose x_0, $z_0 \in Z$ with $x_0 \neq z_0$ and assume without loss that Ω is a filterbase on $Z - \{x_0, z_0\}$. Let X be Z with the topology $\{A \subset X\colon A \cap \{x_0, z_0\} = \varnothing$ or $F \subset A$ for some $F \in \Omega\}$. Let ψ: $X \to Z$ be defined by $\psi(x) = x$ if $x \notin \{x_0, z_0\}$, $\psi(x_0) = z_0$ and $\psi(z_0) = x_0$. Let $x \in X$ and let Ω^* be a filterbase on $X - \{x\}$ with $\Omega^* \to x$. Then $x \in \{x_0, z_0\}$ and we may assume without loss that Ω^* is stronger than Ω. Since $\theta\text{-ad}_s\, \psi(\Omega^*) \subset \text{ad}_\theta\, \Omega = \varnothing$, ψ has a θ-semisubclosed graph. As $\psi(x_0) \neq \psi(z_0)$, $(x_0, z_0) \notin \mathcal{E}(\psi, \psi, X \times X, Z)$. Let $W \in \Sigma((x_0, z_0))$. There are $F, F^* \in \Omega$ with $(F \cup \{x_0\}) \times (F^* \cup \{z_0\}) \subset W$. Let $p \in F \cap F^*$. Then $(p, p) \in W$ and $(p, p) \in \mathcal{E}(\psi, \psi, X \times X, Z)$. Thus $(x_0, z_0) \in \text{cl}(\mathcal{E}(\psi, \psi, X \times X, Z))$ and Z does not satisfy the condition of the theorem. The proof is complete.

2. Some applications of the results of §1. In this section we apply some of the equivalences of S-closedness established in Theorem 2. As our first application we offer a "fixed set theorem" for multifunctions on S-closed spaces. A multifunction ψ from a set X to a set Y is a function ψ: $X \to 2^Y - \{\varnothing\}$, where 2^Y is the family of subsets of Y. If ψ is a multifunction from X to Y we will write $\psi \in \mathfrak{M}(X, Y)$.

THEOREM 4. *Let X be S-closed and let $\psi \in \mathfrak{M}(X, X)$ with the property that $\psi(A)$ is θ-semiclosed when A is θ-semiclosed. Then there is a nonempty θ-semiclosed $K \subset X$ with $\psi(K) = K$.*

PROOF. Let $\Omega = \{A \subset X\colon A \neq \varnothing,\ \psi(A) \subset A$ and A is θ-semiclosed in $X\}$ and apply Zorn's Lemma to choose a minimal element, K, of Ω under set inclusion. It follows easily that $\psi(K) = K$ and the proof is complete.

Let X be a topological space, and Y a partially ordered set (poset). A function ψ: $X \to Y$ is *upper-* (*lower-*) *semicontinuous* if $\psi^{-1}(\{y \in Y\colon y > c\})$ ($\psi^{-1}(\{y \in Y\colon$

$y \lessdot c\}))$ is closed in X for each $c \in Y$. A quasi order $\lessdot$ (reflexive and transitive) on a nonempty topological space X is *upper-* (*lower-*) *semicontinuous* if for each $x \in X$, $\{y \in X: x \lessdot y\}$ ($\{y \in X: y \lessdot x\}$) is closed in X. $\psi \in \mathfrak{M}(X, Y)$ is *upper-* (*lower-*) *semicontinuous* if $\psi^{-1}(F)$ is closed (open) in X for each closed (open) $F \subset Y$. The following theorems are due to Birkhoff **[B]** and Wallace **[W]** for a compact space X.

[B] *Every upper-* (*lower-*) *semicontinuous function from X to a poset assumes a maximal* (*minimal*) *value.*

[W] *X has a maximal* (*minimal*) *element with respect to each upper-* (*lower-*) *semicontinuous quasi order on X.*

Ceder **[Ce]** has proved that X is compact if and only if each upper- (lower-) semicontinuous multifunction from X into the closed subsets of a T_1 space assumes a maximal (minimal) value with respect to set inclusion, and Franklin **[F]** has utilized Ceder's result to show that compactness is equivalent to each of **[B]** and **[W]**. We next apply the results in §2 to produce analogues of **[B]** and **[W]** for S-closed spaces, show that these analogues characterize S-closed spaces, and offer a result for S-closed spaces which is similar to that of Ceder for compact spaces.

A function ψ from a topological space X into a poset Y is *s-upper-* (*lower-*) *semicontinuous* if $\psi^{-1}(\{y \in Y: y \gtrdot c\})$ ($\psi^{-1}(\{y \in Y: y \lessdot c\})$) is θ-semiclosed in X for each $c \in Y$. A quasi order $\lessdot$ on a space X is *s-upper-* (*lower-*) *semicontinuous* if $M(x) = \{y \in X: x \lessdot y\}$ ($L(x) = \{y \in X: y \lessdot x\}$) is θ-semiclosed in X for each $x \in X$. If X and Y are spaces, $\psi \in \mathfrak{M}(X, Y)$ is *s-upper-* (*lower-*) *semicontinuous* if for each closed (open) $F \subset Y$, $\psi^{-1}(F)$ ($X - \psi^{-1}(F)$) is θ-semiclosed in X [$\psi^{-1}(F) = \{x \in X: \psi(x) \cap F \neq \varnothing\}$].

PROPOSITION 2. *A necessary and sufficient condition for $\psi \in \mathfrak{M}(X, Y)$ to be s-upper-semicontinuous is that for each $x \in X$ and $W \in \Sigma(\psi(x))$, there is a regular-closed $R \subset X$ with $x \in R$ and $\psi(R) \subset W$.*

Sufficiency. Suppose F is closed in Y and $x \notin \psi^{-1}(F)$. Then $\psi(x) \subset Y - F$. There is a regular closed set R with $x \in R$ and with $\psi(R) \subset Y - F$. So $\psi(R) \cap F = \varnothing$ and $R \cap \psi^{-1}(F) = \varnothing$, hence $x \notin \theta\text{-cl}_s(\psi^{-1}(F))$.

Necessity. Let $x \in X$ and $W \in \Sigma(\psi(x))$, then $\psi^{-1}(Y - W)$ is θ-semiclosed in X and $x \notin \psi^{-1}(Y - W)$. Thus there is an $R \in \mathrm{RC}(x)$ with $R \cap \psi^{-1}(Y - W) = \varnothing$. So $\psi(R) \cap (Y - W) = \varnothing$ which gives $\psi(R) \subset W$.

THEOREM 5. *The following statements are equivalent for a space X.*

(a) *X is S-closed.*

(b) *X has a maximal element with respect to each s-upper-semicontinuous quasi order on X.*

(c) *Each s-upper-semicontinuous function from X to a poset assumes a maximal value.*

(d) *Each s-upper-semicontinuous multifunction into the subsets of a T_1 space assumes a maximal value with respect to set inclusion.*

Proof that (a) *implies* (b). Let X^* be a chain in X. Then $\Omega = \{M(x): x \in X^*\}$ is a family of θ-semiclosed subsets of X with the finite intersection property. So $\bigcap \Omega \neq \varnothing$. Let $x_0 \in \bigcap \Omega$. Then $x < x_0$ for all $x \in X^*$. So, by Zorn's lemma, X has a maximal element.

Proof that (b) *implies* (c). Let $\psi: X \to Y$ be an s-upper-semicontinuous function to the poset Y. Define $<$ on X by $p < q$ if and only if $\psi(p) < \psi(q)$. Then $<$ is an s-upper-semicontinuous quasi order since $M(x) = \psi^{-1}(\{y \in Y: \psi(x) < y\})$, which is θ-semiclosed in X. Hence X has a maximal element x_0 under $<$. $\psi(x_0)$ is a maximal element in $\psi(X)$.

Proof that (c) *implies* (d). We need to show only that if Y is T_1 and $\psi \in \mathfrak{M}(X, Y)$ is an s-upper-semicontinuous multifunction, then ψ is an s-upper-semicontinuous function into $(2^Y, \subset)$. To this end suppose that $x \notin \psi^{-1}(\{A \subset Y: A \supset C\})$, where $C \subset Y$. There is an element $p \in C - \psi(x)$. So $\psi(x) \subset Y - \{p\}$ which is open since Y is T_1. There is an $R \in \mathrm{RC}(x)$ with $\psi(R) \subset Y - \{p\}$. So $p \in C - \psi(R)$. Hence $R \cap \psi^{-1}(\{A \subset Y: A \supset C\}) = \varnothing$ and $\psi^{-1}(\{A \subset Y: A \supset C\})$ is θ-semiclosed in X.

Proof that (d) *implies* (a). Suppose X is not S-closed. Let $\{x(\alpha): \alpha \in \mu\}$ be a net with no s-accumulation point, where μ is an ordinal. Let μ have the order topology and for each $\lambda \in \mu$, let $V(\lambda) = X - \theta\text{-cl}_s\{x(\alpha): \alpha > \lambda\}$. Then $\{V(\lambda)\}$ covers X. Define $\psi: X \to 2^\mu$ by $\psi(x) = \{\alpha: \alpha < \lambda(x)\}$ where $\lambda(x)$ is the first element λ of μ with $x \in V(\lambda)$. Clearly $\psi(X)$ has no maximal element. Let $x \in X$ and let $W \in \Sigma(\psi(x))$. There is an $R \in \mathrm{RC}(x)$ such that $R \subset V(\lambda(x))$. Let $y \in R$. Then $\lambda(y) < \lambda(x)$, so $\psi(y) \subset \psi(x) \subset W$. Hence $\psi(R) \subset W$ and ψ is s-upper-semicontinuous. This is a contradiction of (d).

The proof of the theorem is complete.

THEOREM 6. *The following statements are equivalent for a space X.*

(a) *X is S-closed.*

(b) *X has a minimal element with respect to each s-lower-semicontinuous quasi order on X.*

(c) *Each s-lower-semicontinuous function from X to a poset assumes a minimal value.*

(d) *Each s-lower-semicontinuous multifunction from X into the closed subsets of a space assumes a minimal value with respect to set inclusion.*

PROOF. The proofs that (a) implies (b) and that (b) implies (c) are similar to those of Theorem 6.

Proof that (c) *implies* (d). Let $\psi \in \mathfrak{M}(X, Y)$ be an s-lower-semicontinuous multifunction into the closed subsets of a space Y. We shall show that $\psi^{-1}(\{A \subset Y: A \subset C\})$ is θ-semiclosed in X for each closed subset C of Y. The result will then follow easily from (c). Suppose $p \notin \psi^{-1}(\{A \subset Y: A \subset C\})$ where C is closed in Y. Then $Y - C$ is open in Y and $\psi(p) \cap (Y - C) \neq \varnothing$. So $p \notin X - \psi^{-1}(Y - C)$ which is θ-semiclosed in X. Let $R \in \mathrm{RC}(p)$ in X with $R \subset \psi^{-1}(Y - C)$. Then we see that for each $x \in R$ we have $\psi(x) - C \neq \varnothing$.

Consequently

$$R \cap \psi^{-1}(\{A \subset Y: A \subset C\}) = \varnothing.$$

So $p \notin \theta\text{-cl}_s(\psi^{-1}(\{A \subset Y: A \subset C\}))$.

Proof that (d) *implies* (a). Suppose X is not S-closed and let $\{x(\alpha): \alpha \in \mu\}$ be a net with no s-accumulation point, where μ is an ordinal. Let μ have the order topology, define $V(\lambda)$ as above for each λ, and define $\psi \in \mathfrak{M}(X, \mu)$ by $\psi(x) = \{\lambda \in \mu: x \in V(\lambda)\}$. Then $\psi(x)$ is closed in Y for each x since $\psi(x) = \{\lambda \in \mu: \lambda > \lambda(x)\}$. We now show that ψ is s-lower-semicontinuous. Let W be open in Y and suppose $x \in \psi^{-1}(W)$. Let $\alpha \in \psi(x) \cap W$. Then $x \in V(\alpha)$. There is a regular-closed $R \subset X$ with $x \in R \subset V(\alpha)$. Let $y \in R$. Then $\alpha \in \psi(y)$ and so $\psi(y) \cap W \neq \varnothing$. Hence $y \in \psi^{-1}(W)$. So $R \subset \psi^{-1}(W)$. This gives $x \notin \theta\text{-cl}_s(X - \psi^{-1}(W))$. Therefore $X - \psi^{-1}(W)$ is θ-semiclosed. Since ψ clearly assumes no minimal value with respect to set inclusion, we reach a contradiction.

References

[A-U] P. Alexandroff and P. Urysohn, *Zur Theorie der topologischen Räume*, Math. Ann. **92** (1924), 258–266.

[B] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., no. 25, Amer. Math. Soc., Providence, R. I., 1948.

[C] D. Cameron, *Properties of S-closed spaces*, Proc. Amer. Math. Soc. **72** (1978), 581–586.

[Ce] J. Ceder, *Compactness and semicontinuous carriers*, Proc. Amer. Math. Soc. **14** (1963), 991–993.

[C-H_1] S. G. Crossley and S. K. Hildebrand, *Semi-closed sets and semi-continuity in topological spaces*, Texas J. Sci. **22** (1971), 123–126.

[C-H_2] ______, *Semi-topological properties*, Fund. Math. **74** (1972), 233–254.

[F] S. Franklin, *Compactness and semi-continuity*, Israel J. Math. **3** (1965), 13–14.

[H_1] R. Herrmann, *RC-convergence* (preprint).

[H_2] ______, *A nonstandard approach to S-closed spaces*, Proc. Amer. Math. Soc. **75** (1979), 311–317.

[H-L_1] L. L. Herrington and P. E. Long, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc. **48** (1975), 469–475.

[H-L_2] ______, *A characterization of minimal Hausdorff spaces*, Proc. Amer. Math. Soc. **57** (1976), 373–374.

[J_1] J. Joseph, *More characterizations of H-closed spaces*, Proc. Amer. Math. Soc. **63** (1977), 160–164.

[J_2] ______, *Characterizations of minimal Hausdorff spaces*, Proc. Amer. Math. Soc. **61** (1976), 145–148.

[J_3] ______, *On Urysohn-closed and minimal Urysohn spaces*, Proc. Amer. Math. Soc. **68** (1978), 235–242.

[K] J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

[L] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[S-S] C. T. Scarborough and A. H. Stone, *Products of nearly compact spaces*, Trans. Amer. Math. Soc. **124** (1966), 131–147.

[T_1] T. Thompson, *S-closed spaces*, Proc. Amer. Math. Soc. **60** (1976), 335–338.

[T_2] ______, *Semicontinuous and irresolute images of S-closed spaces*, Proc. Amer. Math. Soc. **66** (1977), 359–362.

[W] A. D. Wallace, *A fixed point theorem*, Bull. Amer. Math. Soc. **51** (1945), 413–416.

Department of Mathematics, Howard University, Washington, D. C. 20059