THE HOMOTOPY THOM CLASS OF A SPHERICAL FIBRATION

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ABSTRACT. We investigate the following problems. Given a spherical fibration, does the Whitehead square of its homotopy Thom class vanish? If so, is the homotopy Thom class a cyclic homotopy class?

1. Introduction. Let $p : E \to B$ denote a Hurewicz fibration $\xi$ with fiber $F$. Applying the mapping cone construction to the vertical maps in the commutative diagram

$$
\begin{array}{ccc}
F & \subset & E \\
\downarrow & & \downarrow p \\
* & \subset & B
\end{array}
$$

yields a map $\mu : \Sigma F \to T(\xi)$. The Thom space $T(\xi)$ of $\xi$ is the mapping cone of $p$ while $\mu$ is by definition the homotopy Thom class of $\xi$.

We consider only spherical fibrations over locally finite, connected CW-complexes. Let $p : E \to B$ be a fibration $\xi$ whose fiber is homotopy equivalent to $S^{n-1}$. Recall that $T(\xi)$ is then $(n-1)$-connected and $\mu$ generates $\pi_n(T(\xi))$, which is isomorphic to $\mathbb{Z}$ if $p$ is orientable and $\mathbb{Z}/2$ otherwise. Let $\tilde{p} : \tilde{E} \to B$ denote the associated cone fiber space of $\xi$. (See [4, Appendix].) The fiber inclusion of pairs $(CF, F) \subset (E, E)$ induces a map of quotient spaces $CF/F \to \tilde{E}/E$ which we can identify with $\mu$. Let $U$ denote the Thom class in integral cohomology for $\xi$ oriented. Now $\mu$ is dual to $U$ under the Hurewicz isomorphism with respect to the orientation on $CF/F$ induced by $U$. For $\xi$ nonorientable, $\mu$ is clearly dual to the mod 2 Thom class under the mod 2 Hurewicz isomorphism. The homotopy Thom class of an orthogonal vector bundle is defined with reference to the associated sphere bundle.

In this note we investigate the following:

Problem. Given a spherical fibration with homotopy Thom class $\mu$, does the Whitehead square $[\mu, \mu]$ vanish? If so, is $\mu$ a cyclic homotopy class?

Let $\omega_n$ denote the Whitehead square $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ where $\iota_n$ represents the identity map. This problem generalizes the classical problem of the vanishing of $\omega_n$, since $\iota_n$ is the homotopy Thom class of the trivial fibration $p : S^{n-1} \to \ast$. 

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2. Vanishing conditions for $[\mu, \mu]$.

**Proposition 2.1.** Let $p: E \to B$ denote an oriented $(2m - 1)$-spherical fibration $\xi$. If the Euler class $\chi(\xi)$ is divisible by an odd prime in $H^{2m}(B; \mathbb{Z})$, then $[\mu, \mu] \neq 0$. Further, $[\mu, \mu]$ is nontrivial in the rational homotopy of $T(\xi)$ if $\chi(\xi)$ is a torsion class.

**Proof.** Suppose $[\mu, \mu] = 0$ and set $n = 2m$. Thus $\mu: S^n \to T(\xi)$ admits an extension $g: S^n \cup_{\omega} e^{2n} \to T(\xi)$. Let $U$ denote the Thom class of $\xi$ in integral cohomology. Since $S^n \cup_{\omega} e^{2n}$ is the Thom complex of the tangent bundle $\tau(S^n)$ of $S^n$, $g^*U$ is (up to sign) the Thom class for $\tau(S^n)$. Up to sign,

$$g^*(U \cdot \chi(\xi)) = (g^*U)^2 = \chi(S^n) \cdot g^*U = 2(\text{generator}).$$

Thus $U \cdot \chi(\xi)$ and consequently $\chi(\xi)$ via the Thom isomorphism are not divisible by any odd prime.

Suppose that $\chi(\xi)$ is a torsion class. Since the cup product pairing $H^n(T\xi; \mathbb{Z}) \otimes H^n(T\xi; \mathbb{Z}) \to H^{2n}(T\xi; \mathbb{Z})$ is not injective, $[\mu, \mu]$ is not a torsion class in $\pi_{2n-1}(T(\xi))$ by [13].

**Remarks.** (i) It follows from Proposition 2.1 that $[\mu, \mu]$ is nontrivial for any oriented $(2m - 1)$-spherical fibration over $B$ with dimension $B < 2m$.

(ii) The converse to Proposition 2.1 is false. For any integer $n > 1$, consider $\xi = \eta$ over complex projective space $CP^n$ where $\eta$ denotes the complex Hopf line bundle. If $[\mu, \mu] = 0$, then $\Sigma(c \circ h)$ must have order 2 in $\pi_{4n}(\Sigma(CP^{2n-1}/CP^{n-1}))$ where $h: S^{4n-1} \to CP^{2n-1}$ is the Hopf fibration and $c$ denotes the collapsing map. But the $p$-primary component of $\Sigma(c \circ h)$ must be nontrivial for any odd prime $p < n + 1$ such that $p$ does not divide $n + 1$. Thus $[\mu, \mu] \neq 0$ while $\chi(\xi)$ is not divisible by any odd prime.

**Proposition 2.2.** Let $n$ be any odd integer such that $n + 1$ is not a power of 2. Let $p: E \to B$ denote any $(n - 1)$-spherical fibration $\xi$ with dimension $B < n - 2s$ where the positive integer $s$ is defined by $n + 1 \equiv 2s \pmod{2^{s+1}}$. Then $[\mu, \mu]$ has order 2 where $\mu$ denotes the homotopy Thom class of $\xi$. If $\xi$ has trivial Stiefel-Whitney classes and dimension $B < n$, then again $[\mu, \mu]$ is nonzero.

**Proof.** We write $n + 1 = 2^s + 2t$. Expansion of $Sq^2Sq^{2t}$ by the Adem relations and further decompositions of $Sq^j$ for $n - 2^{s-1} < j < n$ yield a relation

$$Sq^2Sq^{2t} + \sum_{i=0}^{s-1} Sq^{2^i} \beta_i = 0$$

on mod 2 classes of dimension $< n$. Here $\beta_i$ is understood to be the trivial operation whenever necessary. Let $\varphi$ denote any nonstable secondary operation associated to the above relation. Suppose either that dimension $B < n - 2^s$ or else that dimension $B < n$ and $\xi$ has trivial Stiefel-Whitney classes. Clearly $\varphi$ is defined on the mod 2 Thom class $U$ of $\xi$ and $\varphi(U)$ vanishes with zero indeterminacy by dimensionality. Recall that $\varphi$ detects $\omega_n$ by [3]; that is, $\varphi$ is nontrivial in the mapping cone of $\omega_n$. So $\mu: S^n \to T(\xi)$ cannot extend to the mapping cone of $\omega_n$ by naturality of $\varphi$. 

Remark. The following example shows the difficulty in obtaining an analogous result whenever \( n + 1 \) is a power of 2 and \( n > 7 \). Let \( \alpha \) denote the real Hopf line bundle over \( S^1 \). Let \( \xi \) denote the sphere bundle of \( \sigma \oplus (n - 1) \) over \( S^1 \). Note that \( T(\xi) = S^n \cup_2 e^{n+1} \). For \( n \) odd and \( j < 2n \), \( 2 \cdot \pi_j(S^n) \) is the kernel of the morphism \( \pi_j(S^n) \to \pi_j(S^n \cup_2 e^{n+1}) \) induced by the inclusion of the bottom cell. Thus \( [\mu, \mu] = 0 \) iff \( \omega_n \in 2 \cdot \pi_{2n-1}(S^n) \). For example, \( \omega_{15} \in 2 \cdot \pi_{29}(S^{15}) \) by [12].

Proposition 2.3. Let \( p: E \to B \) denote an oriented \((n - 1)\)-spherical fibration \( \xi \) over a finite complex \( B \). For \( n \) even, suppose that the reduced integral homology of \( B \) is torsion. Then \( [\mu, \mu] \) has infinite order in \( \pi_{2n-1}(T(\xi)) \). For \( n \) odd, suppose that the reduced integral homology consists of odd torsion. Then \( [\mu, \mu] = 0 \) iff \( n = 1, 3 \) or 7.

Proof. The case \( n \) even is a consequence of Proposition 2.1. For \( n \) odd with \( n > 1 \), the induced map \( \mu(2): S^n(\xi) \to T(\xi)(2) \) on the simply-connected 2-localizations induces an isomorphism on integral homology and so is a homotopy equivalence. Thus \( [\mu, \mu] = \mu_*\omega_n = 0 \) iff \( \omega_n = 0 \).

We have been informed that W. Sutherland has unpublished results on the homotopy Thom class. We thank the referee for his helpful comments. The following two theorems are somewhat related to a conjecture of Mahowald in [9, p. 255].

We recall that the span of a smooth connected manifold \( M \) is the maximum number of linearly independent vector fields on \( M \). A spin manifold is an oriented manifold for which \( w_2 M = 0 \).

Theorem 2.4. Let \( M^n \) be a closed connected oriented smooth manifold with \( n \equiv 1 \) (mod 4). If \( [\mu, \mu] = 0 \) then \( 1 < \text{span} M < 2 \) where \( \mu: S^n \to T(\tau M) \) denotes the homotopy Thom class of the tangent bundle. Let \( \nu \) denote the normal bundle to an embedding of \( M^n \) in \( \mathbb{R}^{2n} \). Then \( [\bar{\mu}, \bar{\nu}] \) has order 2 where \( \bar{\mu}: S^n \to T(\nu) \) denotes the homotopy Thom class.

Proof. We can suppose \( n > 1 \) since \( \text{span} S^1 = 1 \) and \( \mu_*\omega_1 = 0 \). Clearly \( \text{span} M^n = 1 \) if the Stiefel-Whitney class \( w_{n-1} M \neq 0 \). So assume that \( w_{n-1} M = 0 \). By [8] let \( \Phi \) denote the nonstable secondary operation associated to the relation \( \text{Sq}^2\text{Sq}^{n-1} = 0 \) on integral classes of dimension \( < n \) such that

\[
\Phi(U) = U \cdot \left( O(\tau M) + w_2 M \cdot w_{n-2} M \right)
\]

with zero indeterminacy. Here \( U \) denotes the Thom class of \( \tau M \) while \( O(\tau M) \) denotes the unique higher-order obstruction to two linearly independent sections. Now \( \Phi(U) \neq 0 \) since \( [\mu, \mu] = 0 \) by hypothesis and \( \Phi \) detects \( \omega_n \) by [3]. So \( O(\tau M) \neq 0 \) iff \( w_2 M \cdot w_{n-2} M = 0 \). Either \( O(\tau M) \neq 0 \) or \( w_{n-2} M \neq 0 \) so span \( M < 2 \).

Similarly, \( \Phi(U_\nu) \) is defined and vanishes with zero indeterminacy. We recall from [7] that the top cell in the Thom complex \( T(\nu) \) associated to the normal bundle of an embedding in Euclidean space is spherical. Since \( \Phi \) detects \( \omega_n, [\bar{\mu}, \bar{\nu}] = \bar{\mu}_* \omega_n \) must be nontrivial and so has order 2.
Theorem 2.5. Let $M^n$ be a closed connected smooth spin manifold with $n \equiv 3 \pmod{8}$. If $[\mu, \mu] = 0$, then span $M = 3$ where $\mu: S^n \to T(\tau M)$ denotes the homotopy Thom class of $\tau M$. Let $\nu$ denote the normal bundle to an embedding of $M^n$ in $\mathbb{R}^{2n}$. Then $[\bar{\mu}, \bar{\mu}]$ has order 2 for $n > 3$ where $\bar{\mu}$ denotes the homotopy Thom class for $\nu$.

Proof. The case $n = 3$ follows since $M^3$ is parallelizable and $\omega_3 = 0$. Now Atiyah-Dupont [2] proved that span $M^n > 3$. Write $n = 8t + 3$ for positive $t$ and suppose that $w_{n-3}M = 0$. By [10] there exists a nonstable secondary operation $\Omega$ associated to the relation $Sq^4Sq^8' = 0$ on integral classes $x$ of degree $< 8t + 3$ for which $Sq^2x = 0$ such that $\Omega(U) = U \cdot O(\tau M)$ with zero indeterminacy. Here $O(\tau M)$ represents a second-order $k$-invariant to lifting $\tau M$ in the fibration

$$B \text{Spin}(n - 4) \to B \text{Spin}(n).$$

(2.6)

By [3] $\Omega$ detects $\omega_n$. Since $[\mu, \mu]$ vanishes by hypothesis, $\Omega(U)$ must be nontrivial. Thus $O(\tau M) \neq 0$ so span $M = 3$.

Now $\Omega(U_\nu)$ is defined and vanishes with zero indeterminacy since the top cell in $T(\nu)$ is spherical. If $[\bar{\mu}, \bar{\mu}]$ vanishes, then $\Omega(U_\nu)$ must be nontrivial since $\Omega$ detects $\omega_n$ by [3]. Thus $[\bar{\mu}, \bar{\mu}]$ has order 2.

3. Is $\mu$ cyclic? Recall that $\mu$ is cyclic if the map

$$\mu \nabla 1: S^n \vee T(\xi) \to T(\xi)$$

(3.1)

extends to the product $S^n \times T(\xi)$. Equivalently, $\mu$ is cyclic iff $\mu$ belongs to the $n$th evaluation subgroup $G_n(T(\xi))$ of $T(\xi)$. If $\mu$ is cyclic, then $[\mu, \mu] = 0$ by the composite

$$S^n \times S^n \overset{1 \times \mu}{\to} S^n \times T(\xi) \overset{g}{\to} T(\xi)$$

(3.2)

where $g$ extends $\mu \nabla 1$.

If $\mu$ is cyclic for an oriented $(n - 1)$-spherical fibration $\xi$ and $T(\xi)$ is a suspension, Gottlieb showed in [5, Corollary 5-5] that $n = 1, 3$ or 7 and $T(\xi)$ is homotopy equivalent to $S^n$.

Theorem 3.3. Suppose $\mu: S^n \to T(\xi)$ is cyclic for an oriented fibration $p: E \to B$ with $B$ a finite connected complex. If $w_1(\xi)$ is trivial, then $T(\xi)$ is homotopy equivalent to $S^n$ and $n = 1, 3$ or 7. If $w_1(\xi) \neq 0$, then $n$ is odd, the Euler-Poincaré characteristic $\chi(B) = 1$, and the reduced integral homology of $B$ is a vector space over $\mathbb{Z}/2$. Further, $n = 7$ if $\xi$ is orientable with respect to complex $K$-theory.

Proof. By hypothesis $G_n(T(\xi)) = \pi_n(T(\xi))$ so $n$ must be odd by [6, Theorem 1]. Suppose $w_1(\xi) = 0$. Assume that the reduced integral homology of $B$ is nontrivial and let $x$ be a nontrivial cohomology class of smallest positive dimension. Then for any extension $g$ of $\mu \nabla 1$,

$$g^*(U \cdot (x \delta w_{n-1}(\xi))) = g^*(U \cdot Ux) = g^*U \cdot g^*Ux = s_n \otimes Ux + 1 \otimes U \cdot (x \delta w_{n-1}(\xi))$$

(3.4)

where $s_n$ generates $H^n(S^n; \mathbb{Z})$ and $\delta$ denotes the Bockstein-coboundary operator associated to the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$. So $Ux$ and thus $x$ via the
Thom isomorphism have order 2. Since \( x \) was chosen arbitrarily, we may assume \( \rho_2 \neq 0 \). (Here \( \rho_2 \) denotes reduction mod 2.) But \( \rho_2(s_n \otimes Ux) \neq 0 \) in (3.4), a contradiction. (Note that (3.4) uses \( \lambda(Ux) = 0 \) where \( g^*(Ux) = 1 \otimes Ux + s_n \otimes \lambda(Ux) \), but that this fact is not necessary if \( \dim x = n \).) We conclude that \( T(\xi) \) has the homology of \( S^n \). Thus \( T(\xi) \) is homotopy equivalent to \( S^n \) by the argument of [5, Corollary 5-3], since \( T(\xi) \) is a suspension for \( n = 1 \). Finally, \( n = 1, 3 \) or 7 by Proposition 2.3 since \( [\mu, \mu] = 0 \).

Suppose that \( w_n(\xi) \neq 0 \). By [5, Theorem 4-1], \( \chi(T(\xi)) = 0 \). Thus \( \chi(B) = 1 \) since \( \chi(T(\xi)) = 1 + (-1)^n \chi(B) \). Let \( x \in H^i(B; \mathbb{Z}) \) denote any nontrivial class for \( i > 0 \). The calculation in (3.4) yields

\[
g^*(U \cdot (x \delta w_{n-1}(\xi))) = s_n \otimes Ux + s_n \otimes Uz \delta w_{n-1}(\xi) + 1 \otimes U \cdot (x \delta w_{n-1}(\xi))
\]

where \( Uz = \lambda(Ux) \). So \( Ux \) and thus \( x \) must have order 2. Since \( x \) was chosen arbitrarily, the reduced integral homology of \( B \) must be a vector space over \( \mathbb{Z}/2 \).

Finally, we must show that \( n \) must be 7 under the orientability hypothesis. Since \( w_n(\xi) \neq 0 \) and orientability in complex \( K \)-theory implies that \( \delta w_2(\xi) = 0 \), \( n \) must be an odd integer \( \geq 5 \).

Let

\[
S^{2n+1} \xrightarrow{h} \Sigma T(\xi) \xrightarrow{i} Y \rightarrow S^{2n+2} \rightarrow \cdots
\]

denote the Puppe sequence for the map \( h \) obtained by the Hopf construction applied to (3.2). The map in (3.2) induces the trivial morphism on \( H^{2n}(T(\xi); G) \) for any coefficient group \( G \). Consequently, the Hopf invariant of \( h \) is \( \pm 1 \) in integral cohomology (see [11]) and also in complex \( K \)-theory. That is, the free summand of \( K^0(Y) \) is generated by \( y \) and \( x \) where

\[
ch_{n+1}(i^*x) = \Sigma U \text{ in } H^{n+1}(\Sigma T(\xi); \mathbb{Q})
\]

and \( x^2 = \pm y \) in \( K^0(Y)/\text{torsion} \). Equating the coefficients of \( \psi^2 \psi^3 x = \psi^3 \psi^2 x \) yields \( n = 7 \) by the argument of [1].

**References**


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