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THE HOMOTOPY THOM CLASS OF A SPHERICAL FIBRATION

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ABSTRACT. We investigate the following problems. Given a spherical fibration, does the Whitehead square of its homotopy Thom class vanish? If so, is the homotopy Thom class a cyclic homotopy class?

1. Introduction. Let $p\colon E \to B$ denote a Hurewicz fibration ξ with fiber F. Applying the mapping cone construction to the vertical maps in the commutative diagram

$$\begin{array}{ccc} F & \subset & E \\ \downarrow & & \downarrow p \\ * & \subset & B \end{array}$$

yields a map $\mu\colon \Sigma F \to T(\xi)$. The Thom space $T(\xi)$ of ξ is the mapping cone of p while μ is by definition the homotopy Thom class of ξ.

We consider only spherical fibrations over locally finite, connected CW-complexes. Let $p\colon E \to B$ be a fibration ξ whose fiber is homotopy equivalent to S^{n-1}. Recall that $T(\xi)$ is then $(n-1)$-connected and μ generates $\pi_n(T(\xi))$, which is isomorphic to $\mathbf{Z}$ if p is orientable and $\mathbf{Z}/2$ otherwise. Let $\bar{p}\colon \bar{E} \to B$ denote the associated cone fiber space of ξ. (See [**4**, Appendix].) The fiber inclusion of pairs $(CF, F) \subset (\bar{E}, E)$ induces a map of quotient spaces $CF/F \to \bar{E}/E$ which we can identify with μ. Let U denote the Thom class in integral cohomology for ξ oriented. Now μ is dual to U under the Hurewicz isomorphism with respect to the orientation on CF/F induced by U. For ξ nonorientable, μ is clearly dual to the mod 2 Thom class under the mod 2 Hurewicz isomorphism. The homotopy Thom class of an orthogonal vector bundle is defined with reference to the associated sphere bundle.

In this note we investigate the following:

Problem. Given a spherical fibration with homotopy Thom class μ, does the Whitehead square $[\mu, \mu]$ vanish? If so, is μ a cyclic homotopy class?

Let ω_n denote the Whitehead square $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ where ι_n represents the identity map. This problem generalizes the classical problem of the vanishing of ω_n, since ι_n is the homotopy Thom class of the trivial fibration $p\colon S^{n-1} \to *$.

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2. Vanishing conditions for $[\mu, \mu]$.

PROPOSITION 2.1. *Let $p\colon E \to B$ denote an oriented $(2m-1)$-spherical fibration ξ. If the Euler class $\chi(\xi)$ is divisible by an odd prime in $H^{2m}(B; \mathbf{Z})$, then $[\mu, \mu] \neq 0$. Further, $[\mu, \mu]$ is nontrivial in the rational homotopy of $T(\xi)$ if $\chi(\xi)$ is a torsion class.*

PROOF. Suppose $[\mu, \mu] = 0$ and set $n = 2m$. Thus $\mu\colon S^n \to T(\xi)$ admits an extension $g\colon S^n \cup_{\omega_n} e^{2n} \to T(\xi)$. Let U denote the Thom class of ξ in integral cohomology. Since $S^n \cup_{\omega_n} e^{2n}$ is the Thom complex of the tangent bundle $\tau(S^n)$ of S^n, $g^* U$ is (up to sign) the Thom class for $\tau(S^n)$. Up to sign,

$$g^*(U \cdot \chi(\xi)) = (g^* U)^2 = \chi(S^n) \cdot g^* U = 2(\text{generator}).$$

Thus $U \cdot \chi(\xi)$ and consequently $\chi(\xi)$ via the Thom isomorphism are not divisible by any odd prime.

Suppose that $\chi(\xi)$ is a torsion class. Since the cup product pairing $H^n(T\xi; \mathbf{Z}) \otimes H^n(T\xi; \mathbf{Z}) \to H^{2n}(T\xi; \mathbf{Z})$ is not injective, $[\mu, \mu]$ is not a torsion class in $\pi_{2n-1}(T(\xi))$ by [13].

REMARKS. (i) It follows from Proposition 2.1 that $[\mu, \mu]$ is nontrivial for any oriented $(2m-1)$-spherical fibration over B with dimension $B < 2m$.

(ii) The converse to Proposition 2.1 is false. For any integer $n > 1$, consider $\xi = n\eta$ over complex projective space CP^n where η denotes the complex Hopf line bundle. If $[\mu, \mu] = 0$, then $\Sigma(c \circ h)$ must have order 2 in $\pi_{4n}(\Sigma(CP^{2n-1}/CP^{n-1}))$ where $h\colon S^{4n-1} \to CP^{2n-1}$ is the Hopf fibration and c denotes the collapsing map. But the p-primary component of $\Sigma(c \circ h)$ must be nontrivial for any odd prime $p \leqslant n+1$ such that p does not divide $n+1$. Thus $[\mu, \mu] \neq 0$ while $\chi(\xi)$ is not divisible by any odd prime.

PROPOSITION 2.2. *Let n be any odd integer such that $n+1$ is not a power of 2. Let $p\colon E \to B$ denote any $(n-1)$-spherical fibration ξ with dimension $B \leqslant n - 2^s$ where the positive integer s is defined by $n + 1 \equiv 2^s \pmod{2^{s+1}}$. Then $[\mu, \mu]$ has order 2 where μ denotes the homotopy Thom class of ξ. If ξ has trivial Stiefel-Whitney classes and dimension $B < n$, then again $[\mu, \mu]$ is nonzero.*

PROOF. We write $n + 1 = 2^s + 2t$. Expansion of $Sq^{2^s}Sq^{2t}$ by the Adem relations and further decompositions of Sq^j for $n - 2^{s-1} < j \leqslant n$ yield a relation

$$Sq^{2^s}Sq^{2t} + \sum_{i=0}^{s-1} Sq^{2^i}\beta_i = 0$$

on mod 2 classes of dimension $\leqslant n$. Here β_i is understood to be the trivial operation whenever necessary. Let φ denote any nonstable secondary operation associated to the above relation. Suppose either that dimension $B \leqslant n - 2^s$ or else that dimension $B < n$ and ξ has trivial Stiefel-Whitney classes. Clearly φ is defined on the mod 2 Thom class U of ξ and $\varphi(U)$ vanishes with zero indeterminacy by dimensionality. Recall that φ detects ω_n by [3]; that is, φ is nontrivial in the mapping cone of ω_n. So $\mu\colon S^n \to T(\xi)$ cannot extend to the mapping cone of ω_n by naturality of φ.

REMARK. The following example shows the difficulty in obtaining an analogous result whenever $n + 1$ is a power of 2 and $n > 7$. Let α denote the real Hopf line bundle over S^1. Let ξ denote the sphere bundle of $\sigma \oplus (n - 1)$ over S^1. Note that $T(\xi) = S^n \cup_2 e^{n+1}$. For n odd and $j < 2n$, $2 \cdot \pi_j(S^n)$ is the kernel of the morphism $\pi_j(S^n) \to \pi_j(S^n \cup_2 e^{n+1})$ induced by the inclusion of the bottom cell. Thus $[\mu, \mu] = 0$ iff $\omega_n \in 2 \cdot \pi_{2n-1}(S^n)$. For example, $\omega_{15} \in 2 \cdot \pi_{29}(S^{15})$ by [**12**].

PROPOSITION 2.3. *Let p: $E \to B$ denote an oriented $(n - 1)$-spherical fibration ξ over a finite complex B. For n even, suppose that the reduced integral homology of B is torsion. Then $[\mu, \mu]$ has infinite order in $\pi_{2n-1}(T(\xi))$. For n odd, suppose that the reduced integral homology consists of odd torsion. Then $[\mu, \mu] = 0$ iff $n = 1, 3$ or 7.*

PROOF. The case n even is a consequence of Proposition 2.1. For n odd with $n > 1$, the induced map $\mu_{(2)}$: $S^n_{(2)} \to T(\xi)_{(2)}$ on the simply-connected 2-localizations induces an isomorphism on integral homology and so is a homotopy equivalence. Thus $[\mu, \mu] = \mu_* \omega_n = 0$ iff $\omega_n = 0$.

We have been informed that W. Sutherland has unpublished results on the homotopy Thom class. We thank the referee for his helpful comments. The following two theorems are somewhat related to a conjecture of Mahowald in [**9**, p. 255].

We recall that the span of a smooth connected manifold M is the maximum number of linearly independent vector fields on M. A spin manifold is an oriented manifold for which $w_2 M = 0$.

THEOREM 2.4. *Let M^n be a closed connected oriented smooth manifold with $n \equiv 1$ (mod 4). If $[\mu, \mu] = 0$ then $1 \leqslant \operatorname{span} M \leqslant 2$ where μ: $S^n \to T(\tau M)$ denotes the homotopy Thom class of the tangent bundle. Let ν denote the normal bundle to an embedding of M^n in $\mathbf{R}^{2n}$. Then $[\bar{\mu}, \bar{\mu}]$ has order 2 where $\bar{\mu}$: $S^n \to T(\nu)$ denotes the homotopy Thom class.*

PROOF. We can suppose $n > 1$ since $\operatorname{span} S^1 = 1$ and $\mu_* \omega_1 = 0$. Clearly $\operatorname{span} M^n = 1$ if the Stiefel-Whitney class $w_{n-1} M \neq 0$. So assume that $w_{n-1} M = 0$. By [**8**] let Φ denote the nonstable secondary operation associated to the relation $\mathrm{Sq}^2 \mathrm{Sq}^{n-1} = 0$ on integral classes of dimension $\leqslant n$ such that

$$\Phi(U) = U \cdot (O(\tau M) + w_2 M \cdot w_{n-2} M)$$

with zero indeterminacy. Here U denotes the Thom class of τM while $O(\tau M)$ denotes the unique higher-order obstruction to two linearly independent sections. Now $\Phi(U) \neq 0$ since $[\mu, \mu] = 0$ by hypothesis and Φ detects ω_n by [**3**]. So $O(\tau M) \neq 0$ iff $w_2 M \cdot w_{n-2} M = 0$. Either $O(\tau M) \neq 0$ or $w_{n-2} M \neq 0$ so $\operatorname{span} M \leqslant 2$.

Similarly, $\Phi(U_\nu)$ is defined and vanishes with zero indeterminacy. We recall from [**7**] that the top cell in the Thom complex $T(\nu)$ associated to the normal bundle of an embedding in Euclidean space is spherical. Since Φ detects ω_n, $[\bar{\mu}, \bar{\mu}] = \bar{\mu}_* \omega_n$ must be nontrivial and so has order 2.

THEOREM 2.5. *Let M^n be a closed connected smooth spin manifold with $n \equiv 3$ (mod 8). If $[\mu, \mu] = 0$, then* span $M = 3$ *where $\mu\colon S^n \to T(\tau M)$ denotes the homotopy Thom class of τM. Let ν denote the normal bundle to an embedding of M^n in $\mathbf{R}^{2n}$. Then $[\bar{\mu}, \bar{\mu}]$ has order 2 for $n > 3$ where $\bar{\mu}$ denotes the homotopy Thom class for ν.*

PROOF. The case $n = 3$ follows since M^3 is parallelizable and $\omega_3 = 0$. Now Atiyah-Dupont [2] proved that span $M^n \geqslant 3$. Write $n = 8t + 3$ for positive t and suppose that $w_{n-3}M = 0$. By [10] there exists a nonstable secondary operation Ω associated to the relation $\mathrm{Sq}^4\mathrm{Sq}^{8t} = 0$ on integral classes x of degree $\leqslant 8t + 3$ for which $\mathrm{Sq}^2 x = 0$ such that $\Omega(U) = U \cdot O(\tau M)$ with zero indeterminacy. Here $O(\tau M)$ represents a second-order k-invariant to lifting τM in the fibration

$$B\,\mathrm{Spin}(n-4) \to B\,\mathrm{Spin}(n). \tag{2.6}$$

By [3] Ω detects ω_n. Since $[\mu, \mu]$ vanishes by hypothesis, $\Omega(U)$ must be nontrivial. Thus $O(\tau M) \neq 0$ so span $M = 3$.

Now $\Omega(U_\nu)$ is defined and vanishes with zero indeterminacy since the top cell in $T(\nu)$ is spherical. If $[\bar{\mu}, \bar{\mu}]$ vanishes, then $\Omega(U_\nu)$ must be nontrivial since Ω detects ω_n by [3]. Thus $[\bar{\mu}, \bar{\mu}]$ has order 2.

3. Is μ cyclic? Recall that μ is cyclic if the map

$$\mu \nabla 1\colon S^n \vee T(\xi) \to T(\xi) \tag{3.1}$$

extends to the product $S^n \times T(\xi)$. Equivalently, μ is cyclic iff μ belongs to the nth evaluation subgroup $G_n(T(\xi))$ of $T(\xi)$. If μ is cyclic, then $[\mu, \mu] = 0$ by the composite

$$S^n \times S^n \xrightarrow{1\times\mu} S^n \times T(\xi) \xrightarrow{g} T(\xi) \tag{3.2}$$

where g extends $\mu\nabla 1$.

If μ is cyclic for an oriented $(n-1)$-spherical fibration ξ and $T(\xi)$ is a suspension, Gottlieb showed in [5, Corollary 5-5] that $n = 1$, 3 or 7 and $T(\xi)$ is homotopy equivalent to S^n.

THEOREM 3.3. *Suppose $\mu\colon S^n \to T(\xi)$ is cyclic for an oriented fibration $p\colon E \to B$ with B a finite connected complex. If $w_n(\xi)$ is trivial, then $T(\xi)$ is homotopy equivalent to S^n and $n = 1$, 3 or 7. If $w_n(\xi) \neq 0$, then n is odd, the Euler-Poincaré characteristic $\chi(B) = 1$, and the reduced integral homology of B is a vector space over $\mathbf{Z}/2$. Further, $n = 7$ if ξ is orientable with respect to complex K-theory.*

PROOF. By hypothesis $G_n(T(\xi)) = \pi_n(T(\xi))$ so n must be odd by [6, Theorem 1]. Suppose $w_n(\xi) = 0$. Assume that the reduced integral homology of B is nontrivial and let x be a nontrivial cohomology class of smallest positive dimension. Then for any extension g of $\mu\nabla 1$,

$$\begin{aligned} g^*(U\cdot(x\delta w_{n-1}(\xi))) &= g^*(U\cdot Ux) = g^*U\cdot g^*Ux \\ &= s_n \otimes Ux + 1 \otimes U\cdot(x\delta w_{n-1}(\xi)) \end{aligned} \tag{3.4}$$

where s_n generates $H^n(S^n; \mathbf{Z})$ and δ denotes the Bockstein-coboundary operator associated to the coefficient sequence $\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2$. So Ux and thus x via the

Thom isomorphism have order 2. Since x was chosen arbitrarily, we may assume $\rho_2 x \neq 0$. (Here ρ_2 denotes reduction mod 2.) But $\rho_2(s_n \otimes Ux) \neq 0$ in (3.4), a contradiction. (Note that (3.4) uses $\lambda(Ux) = 0$ where $g^*(Ux) = 1 \otimes Ux + s_n \otimes \lambda(Ux)$, but that this fact is not necessary if $\dim x = n$.) We conclude that $T(\xi)$ has the homology of S^n. Thus $T(\xi)$ is homotopy equivalent to S^n by the argument of [5, Corollary 5-3], since $T(\xi)$ is a suspension for $n = 1$. Finally, $n = 1$, 3 or 7 by Proposition 2.3 since $[\mu, \mu] = 0$.

Suppose that $w_n(\xi) \neq 0$. By [5, Theorem 4-1], $\chi(T(\xi)) = 0$. Thus $\chi(B) = 1$ since $\chi(T(\xi)) = 1 + (-1)^n\chi(B)$. Let $x \in H^i(B; \mathbf{Z})$ denote any nontrivial class for $i > 0$. The calculation in (3.4) yields

$$g^*(U \cdot (x\delta w_{n-1}(\xi))) = s_n \otimes Ux + s_n \otimes Uz\delta w_{n-1}(\xi) + 1 \otimes U \cdot (x\delta w_{n-1}(\xi)) \tag{3.5}$$

where $Uz = \lambda(Ux)$. So Ux and thus x must have order 2. Since x was chosen arbitrarily, the reduced integral homology of B must be a vector space over $\mathbf{Z}/2$.

Finally, we must show that n must be 7 under the orientability hypothesis. Since $w_n(\xi) \neq 0$ and orientability in complex K-theory implies that $\delta w_2(\xi) = 0$, n must be an odd integer $\geqslant 5$.

Let

$$S^{2n+1} \xrightarrow{h} \Sigma T(\xi) \xrightarrow{i} Y \rightarrow S^{2n+2} \rightarrow \cdots$$

denote the Puppe sequence for the map h obtained by the Hopf construction applied to (3.2). The map in (3.2) induces the trivial morphism on $H^{2n}(T(\xi); G)$ for any coefficient group G. Consequently, the Hopf invariant of h is ± 1 in integral cohomology (see [11]) and also in complex K-theory. That is, the free summand of $\tilde{K}^0(Y)$ is generated by x and y where

$$ch_{n+1}(i^*x) = \Sigma U \quad \text{in } H^{n+1}(\Sigma T(\xi); Q)$$

and $x^2 = \pm y$ in $\tilde{K}^0(Y)/\text{torsion}$. Equating the coefficients of $\psi^2\psi^3 x = \psi^3\psi^2 x$ yields $n = 7$ by the argument of [1].

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