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SPECIAL HANDLEBODY DECOMPOSITIONS OF SIMPLY-CONNECTED ALGEBRAIC SURFACES

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ABSTRACT. In this article we prove that any nonsingular complete-intersection surface admits a handlebody decomposition with no 1- and 3-handles. This generalizes results of Rudolph, Harer and Akbuluf on hypersurfaces of $\mathbf{CP}^3$.

Introduction. Among the problems posed at the Stanford Conference (24th Summer Research Institute, August, 1976) is the following [**K**, Preliminary List].

Problem 50 (*Kirby*). Does every simply-connected closed 4-manifold have a handlebody decomposition without 1-handles? Without 1- and 3-handles?

Rudolph [**R**] shows that any nonsingular hypersurface of $\mathbf{CP}^3$ has a handlebody decomposition without 1-handles (or dually, without 3-handles), however, Rudolph's method does not allow one to eliminate both 1- and 3-handles simultaneously. In December, 1976, at a lecture at the Institute for Advanced Study, Kirby exhibited a handlebody decomposition of the Kummer surface without both 1- and 3-handles. See [**HKK**].

The Kummer surface is diffeomorphic to a nonsingular quartic in $\mathbf{CP}^3$ and we sketch here the following generalization of the [**HKK**] result exhibited by Kirby.

THEOREM. *Suppose X is a nonsingular complete intersection of k distinct hypersurfaces $V_1, \ldots, V_k$ in $\mathbf{CP}^{k+2}$. Then X has a handlebody decomposition without* 1- *and* 3-*handles*.

1. Lefschetz fibrations.

DEFINITION. Let V be an algebraic surface and suppose L is a pencil of curves on V. Then we shall say L is a Lefschetz pencil if and only if

(1) the generic element of L is nonsingular and irreducible.

(2) L has only a finite number of singular elements, each of which has only one ordinary double point as its singularity.

We recall [**W**], [**Z**] that every algebraic surface admits Lefschetz pencils. (For more details on Lefschetz pencils see especially [**AF**], [**W**].)

Furthermore the Lefschetz pencil L gives rise to a rational map f of V to $\mathbf{CP}^1$. If this map is a morphism $f\colon V \to \mathbf{CP}^1$ we shall call L a Lefschetz fibration. It is clear that every Lefschetz pencil on V gives rise to a Lefschetz fibration $\tilde{f}\colon \tilde{V} \to \mathbf{CP}^1$ of $\tilde{V} = \{V$ blown up at the base points of $L\}$ onto $\mathbf{CP}^1$. (See [**AF**].)

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Now suppose $f: V \to \mathbf{CP}^1$ is a Lefschetz fibration and suppose $a_1, \ldots, a_n \in \mathbf{CP}^1 - \{0, \infty\}$ are its critical values. Then for an appropriate choice of paths λ_i in $\mathbf{CP}^1$ connecting 0 to a_i we can define as in **[Z**, p. 135], **[MM1]** Lefschetz vanishing cycles δ_i on $V_0 = f^{-1}(0)$ and Lefschetz relative cycles $\Delta_i \subset f^{-1}(\lambda_i) \subset V$.

We have the following theorem **[AF]** relating the topology of V and V_0.

THEOREM 1. *Let $V_\infty = f^{-1}(\infty)$. Then $V - V_\infty$ has the homotopy type of V_0 with n 2-discs Δ_i attached along the δ_i.*

In fact it is easy to see that we have a handlebody description of V by means of its decomposition into V_0, the Δ_i, V_∞.

We now formulate:

DEFINITION. Let $f: V \to \mathbf{CP}^1$ be a Lefschetz fibration and suppose that the generic fiber of f has genus g. Suppose further there exist a system of paths in $\mathbf{CP}^1$ such that we can define Lefschetz vanishing and relative cycles δ_i on V_0, Δ_i in V with the following property.

Property W. There exists a subsystem $\delta_1, \ldots, \delta_{2g}$ of Lefschetz vanishing cycles such that $B = \bigcup_{i=1}^{i=2g} \delta_i$ is a bouquet of $2g$ circles with $V_0 - B$ homeomorphic to a 2-disc.

Then we shall call $f: V \to \mathbf{CP}^1$ an exceptional Lefschetz fibration. We note the following result from **[Wb]**.

THEOREM 2. *Let $n_0, n_1, \ldots, n_k$ be positive integers. There exists an algebraic curve $V_0 \subset \mathbf{CP}^2$ of degree $n = \prod_{i=0}^{k} n_i$ such that*

1. *V_0 has $n_0 \cdot n_1 \cdot \ldots \cdot n_i \cdot (n_i - 1)$ ordinary singular points of order $n_{i+1} \cdot n_{i+2} \cdot \ldots \cdot n_k$ for $i = 0, 1, \ldots, k-1$ and no other singular points.*

2. *There exist closed discs $D_1, D_2, \ldots, D_{2g}$ in $\widetilde{\mathbf{CP}^2}$ (where g denotes the genus of V_0), such that $D_i \cap V_0 = \partial D_i$ and $D_i \cap D_j = P$ for $i \neq j$ (where P is a fixed point on V_0), and D_i does not meet singular points of V_0, $i, j = 1, 2, \ldots, 2g$.*

3. *Let $\mathbf{CP}^2$ be $\mathbf{CP}^2$ blown up by σ-processes at the singular points of V_0. Let $\tilde{A}$ denote the proper image of A on $\mathbf{CP}^2$ for any $A \subset \mathbf{CP}^2$. Then the bouquet of 1-spheres $\tilde{M} = \bigcup_{i=1}^{2g} \partial \tilde{D}_i$ generates the fundamental group $\pi_1(\tilde{V}_0, \tilde{P})$ and its complement $\tilde{V}_0 - \tilde{M}$ is homeomorphic to an open disc.*

We note that the proof of Theorem 2 actually implies

PROPOSITION 3. *Let V be a nonsingular complete intersection surface in some projective space $\mathbf{P}^N$. Let H_V be a hyperplane section of V and suppose $H_V^2 = m$. Then for every integer n there exists an exceptional Lefschetz fibration $f_n: V_n \to \mathbf{CP}^1$ of $V_n = V$ blown up at mn^2 points, with fiber the proper transform of a hypersurface section of V of degree n.*

2. Handlebody decompositions.

DEFINITION. Suppose $f_1: V_1 \to S^2$, $f_2: V_2 \to S^2$ are Lefschetz fibrations of curves of genus g.

Let a_1, a_2 be regular values of f_1, f_2 resp. and let D_1, D_2 be discs around a_1, a_2 not containing any critical values.

Let $h\colon \partial D_1 \to \partial D_2$ be an orientation reversing diffeomorphism and identifying $f_i^{-1}(D_i)$ with $D_i \times F$ for $i = 1, 2$ (where F is a nonsingular curve of genus g) extend h to $g\colon f_1^{-1}(D_1) \to f_2^{-1}(D_2)$ by the identity on the second factor.

Let

$$V = \overline{V_1 - f_1^{-1}(D_1)} \cup_g \overline{V_2 - f_2^{-1}(D_2)}, \qquad S = \overline{S^2 - D_1} \cup_h \overline{S^2 - D_2}$$

and define $f\colon V \to S$ by

$$f|\overline{V_i - f_i^{-1}(D_i)} = f_i|\overline{V_i - f_i^{-1}(D_i)}, \qquad i = 1, 2.$$

It is clear that $S \approx S^2$ and $f\colon V \to S$ is a Lefschetz fibration of curves of genus g (provided V is algebraic. In all our applications we shall know a priori that V is algebraic.) We write V as $V = V_1 \oplus V_2$ and call it the direct Lefschetz sum of V_1 and V_2. We then have

PROPOSITION 4. *Let $f_i\colon V_i \to S^2$ be exceptional Lefschetz fibrations of curves of the same genus g for $i = 1, 2$.*

Let $V = V_1 \oplus V_2$ and $f\colon V \to S^2$ be defined as above. Then V admits a handlebody decomposition with no 1- *or* 3-*handles.*

PROOF. Since the f_i are exceptional Lefschetz fibrations of genus g there exist systems of paths in $\mathbf{CP}^1$ such that we can define Lefschetz vanishing and relative cycles δ_j^i on $V_{i,0}$ and Δ_j^i on V, $i = 1, j = 1, \ldots, N_1$; $i = 2, j = 1, \ldots, N_2$, such that there exist subsystems δ_j^i, $i = 1, 2$; $j = 1, \ldots, 2g$, of the Lefschetz vanishing cycles satisfying Property W.

Let $\mathcal{V}_i = V_{i,0} \cup (\bigcup_{j=1}^{2g} \Delta_j^i)$, $i = 1, 2$. Then $V_i' = V_i - T(V_{i,\infty})$ admits the following decomposition.

$$V_i' \approx N(V_{i,0}) \cup \bigcup_{j=1}^{N_i} N(\Delta_j^i) = N(\mathcal{V}_i) \cup \bigcup_{j=2g+1}^{N_i} N(\Delta_j^i)$$

where $\approx$ means "deformation retract", $N(A)$ is a regular nbhd of $A \subset V_i'$ in V_i' and $T(X)$ is a tubular neighborhood of X.

Then by Property W we deduce that $N(\mathcal{V}_i)$ is homeomorphic to $D^4 \cup \{2\text{-handle}\}$. In particular let $p_i \in \overline{V_{i,0} - \bigcup_{j=1}^{2g} \delta_j^i}$. Then there exists a 2-disc $D_i \ni p_i$ in $V_{i,0}$ and we have that $H_0^i = N((\overline{V_{i,0} - D_i}) \cup (\bigcup_{j=1}^{2g} \Delta_j^i))$ is a 0-handle while $N(D_i)$, $N(\Delta_j^i)$, $i = 1$; $j = 2g + 1, \ldots, N_1$, $i = 2$; $j = 2g + 1, \ldots, N_2$, are 2-handles attached to H_0^i along $N(D_i) \cap \partial H_0^i$, $N(\Delta_j^i) \cap \partial H_0^i$, respectively.

Thus V_i' admits a handlebody decomposition consisting only of a 0-handle and $N_i - 2g + 1$ 2-handles.

In particular then by duality we obtain that since $V = V_1' \cup V_2'$ then V admits a handlebody decomposition consisting only of a 0-handle, $N_1 + N_2 + 2 - 4g$ 2-handles and a 4-handle as desired.

We now have

THEOREM 5. *Suppose X is a nonsingular complete intersection of k distinct hypersurfaces $V_1, \ldots, V_k$ in $\mathbf{CP}^{k+2}$. Then X has a handlebody decomposition without* 1- *and* 3-*handles.*

PROOF. Since X is a complete intersection we may, using Corollary 6.2 of [MM2], assume that there exist k hypersurfaces $V_i(n_i)$ of degree n_i in $\mathbf{CP}^{k+2}$ such that setting $Y = \bigcap_{i=1}^{k-1} V_i(n_i)$ we have that Y is nonsingular and Y intersects $V_k(n_k)$ transversely. Furthermore, as our theorem is obvious for $\mathbf{CP}^2$ we may also assume without loss of generality that $n_k > 2$. Using Corollary 6.2 of [MM2] we see that there exists a hypersurface V_k' of degree $n_k - 1$ in $\mathbf{CP}^{k+2}$ and a hyperplane H in $\mathbf{CP}^{k+2}$ such that V_k', H intersect Y transversely and V_k', H, V_k have normal crossing in $\mathbf{CP}^{k+2}$.

Let $X' = Y \cap V_k'$, $X'' = Y \cap H$, $C = Y \cap V_k' \cap H$, $m = X \cdot V_k' \cdot H$, $m_1 = X' \cdot H^2$, $m_2 = (X')^2 \cdot H$ (where $\cdot$ is multiplication of cycles in the homology ring of $\mathbf{CP}^{k+2}$ and we identify $H_0(\mathbf{CP}^{k+2})$ with $\mathbf{Z}$).

Then again using Corollary 6.2 of [MM2], we obtain $X = \sigma_m(X') - T(\overline{C}) \cup X'' - T(C)$ where $\sigma_m(X')$ is X' blown up at m points and $T(\overline{C})$, $T(C)$ are tubular neighborhoods of $\overline{C}$ = (strict image of C in $\sigma_m(X')$), and C respectively. But using [P, Chapter 2, Corollary 3], we can obtain $X = \sigma_{m_1}(X') - T(C') \cup \sigma_{m_2}(X'') - T(C'')$ where C', C'' denote strict images.

But using Proposition 3 we see that $\sigma_{m_1}(X')$ and $\sigma_{m_2}(X'')$ both admit exceptional Lefschetz fibrations with C', resp. C'' a typical fiber. Thus we obtain that $X = \sigma_{m_1}(X') \oplus \sigma_{m_2}(X'')$ and so by Proposition 4 we have that X admits the requisite handlebody techniques.

Using similar techniques together with the methods of [M] we can also obtain

THEOREM 6. *Suppose X is either a nonsingular simply-connected elliptic surface with no more than one multiple fiber or a cyclic branched cover of a nonsingular complete intersection surface with branch locus a nonsingular complete intersection curve.*

Then X admits a handlebody decomposition without 1- *and* 3-*handles.*

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