A TOPOLOGICAL CHARACTERIZATION OF A CLASS OF CARDINALS

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Abstract. Let \( m \) be the first measurable cardinal. We say that a cardinal \( \alpha \) is Ulam-stable if, on the discrete space \( D(\alpha) \) of cardinal \( \alpha \), every filter with \( m \)-intersection property can be extended to an ultrafilter with \( m \)-intersection property. The main result we prove is the following: \( \alpha \) is Ulam-stable if and only if its Hewitt-Nachbin realcompletion \( vD(\alpha) \) is paracompact.

0. Introduction. Until now, various classes of cardinals have been considered in mathematical literature (see [1], [6]); a certain amount of attention has been devoted to characterize, in topological terms, some set-theoretic relations connected with them.

Keisler and Tarski [6] introduced a binary relation \( \Re \), in the class of all cardinals, which Comfort-Negrepontis in their paper [1] modified as follows: \( \alpha \not\in \beta \) provided that there is, on the discrete space of cardinal \( \beta \), an \( \alpha \)-complete filter that cannot extend to an \( \alpha \)-complete ultrafilter. In the paper referred to, such set-theoretic relation is described from a topological point of view. The aim of the present paper goes along these lines: we find a topological characterization for the class of all cardinals which do not satisfy the above relation for fixed \( \alpha = m \) (first Ulam-measurable cardinal). We call them Ulam-stable cardinals and denote their class by \( \mathfrak{S}_{m} \). It is well known that if \( \alpha \) is a nonmeasurable cardinal (in the sense of [5]) then the discrete space \( D(\alpha) \) is realcompact, and thus it coincides with its Hewitt-Nachbin realcompletion \( vD(\alpha) \), which is obviously paracompact.

Assuming the existence of an Ulam-measurable cardinal, the problem we solve here is to determine the class of all cardinals \( \xi \) for which \( vD(\xi) \) is paracompact. We prove that \( vD(\xi) \) is paracompact if and only if \( \xi \) is Ulam-stable, i.e., \( \xi \) belongs to \( \mathfrak{S}_{m} \).

1. Basic notions and results. We denote ordinal numbers by Greek letters; a cardinal number is an initial ordinal. We indicate by \( \nu \) the first infinite cardinal and by \( \alpha^{+} \) the smallest cardinal greater than \( \alpha \). A cardinal \( \alpha \) is said to be regular if it cannot be expressed as the sum of fewer than \( \alpha \) cardinals each smaller than \( \alpha \). The discrete space of cardinal \( \beta \) is denoted by \( D(\beta) \). A filter \( \mathcal{F} \) on \( D(\beta) \) has the \( \alpha \)-intersection property (abbr. \( \alpha \text{-i.p.} \)) if \( \bigcap B \neq \emptyset \) whenever \( B \subseteq \mathcal{F} \) and \( |B| < \alpha \); moreover, if \( \bigcap B \in \mathcal{F} \) for \( B \subseteq \mathcal{F} \) and \( |B| < \alpha \), then \( \mathcal{F} \) is said to be \( \alpha \)-complete. An ultrafilter with \( \alpha \text{-i.p.} \) is clearly \( \alpha \)-complete. We recall here that a cardinal \( \alpha \) is

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(Ulam)-measurable if there exists a nonprincipal $v^+$-complete ultrafilter on $D(\alpha)$; let $m$ denote the first measurable cardinal. It is known that if $\alpha > m$ then $\alpha$ is a measurable cardinal.

Furthermore, $D(\alpha)$ is not realcompact and its Hewitt-Nachbin realcompletion $vD(\alpha)$ can be thought of as the space of all $m$-complete ultrafilters on $D(\alpha)$. Other equivalent definitions for measurable cardinals (involving measures) are to be found in [3] and [5]. We say that a cardinal $\alpha$ is $m$-stable (or Ulam-stable) if, on $D(\alpha)$, every filter with m.i.p. extends to an $m$-complete ultrafilter. We say that $\alpha$ is strongly measurable if $\alpha$ is regular and every filter on $D(\alpha)$ having the a.i.p. extends to an $\alpha$-complete ultrafilter. We observe that if $\alpha$ is measurable, every $m$-complete filter on $D(\alpha)$ extends to an $m$-complete ultrafilter if and only if every filter with m.i.p. does.

Furthermore, we need some topological notions and results. Let $X$ be a Tychonoff space; $M$ a dense subset of $X$.

We denote by $\mathcal{F}_M$ the class of all mappings of $M$, taking values in a metric space, which can be continuously extended to $X$. If $\mathcal{F}$ is a filter on $M$, we denote by $\mathcal{F}$ the extension of $\mathcal{F}$ to $X$, namely the filter on $X$ generated by $\mathcal{F}$. We state here without proof two theorems.

**Theorem A (Corson [4]).** For a Tychonoff space $X$ the following are equivalent:

(I) $X$ is paracompact;

(II) If $\mathcal{F}$ is a filter on $X$ such that the image of $\mathcal{F}$ has a cluster point in any metric space into which $X$ is continuously mapped, then $\mathcal{F}$ has a cluster point in $X$.

The second result we quote here was obtained jointly by W. W. Comfort and Teklehaimanot Retta and communicated to me February 1, 1977.

**Theorem B (Comfort-Retta).** If $vD(m)$ is paracompact, then $m$ is strongly measurable.

**Corollary.** Suppose $\alpha > m$. If $vD(\alpha)$ is paracompact, then $m$ is strongly measurable.

2. **Lemmas.** We need two lemmas. The first is a strengthened version of Theorem A.

**Lemma 1.** Let $X$ be a Tychonoff space and $M$ a dense subset of $X$. Then the following are equivalent:

(I) $X$ is paracompact;

(II) If $\mathcal{F}$ is a filter on $M$ such that $f(\mathcal{F})$ has a cluster point for each $f \in \mathcal{F}_M$, then the extension $\mathcal{F}$ of $\mathcal{F}$ to $X$ has a cluster point in $X$.

**Proof.** (I) $\rightarrow$ (II) is obvious. For the converse, let $\mathcal{F}$ be a filter on $X$ such that the image of $\mathcal{F}$ has a cluster point in any metric space into which $X$ is continuously mapped: we must show that $\mathcal{F}$ clusters in $X$. Take the n.b.d. filter $\mathcal{U}(\mathcal{F})$ and its trace $\mathcal{G}$ on $M$. It can be easily checked that $f(\mathcal{G})$ has a cluster point for each $f \in \mathcal{F}_M$ and thus $\mathcal{G}$ clusters in $X$ by the hypothesis.
Consequently, $\mathcal{F}$ has a cluster point in $X$ and so $X$ is paracompact applying Theorem A.

**Lemma 2.** Let $\alpha, \beta$ be infinite cardinals with $\beta > \alpha$ and $\mathcal{F}$ a filter on $D(\beta)$ which fails to have the $\alpha$-i.p. Then there is a cardinal $\gamma < \alpha$ and a family $(A_\xi)_{\xi < \gamma}, A_\xi \in \mathcal{F}$ with the following properties:

(a) $\bigcap_{\xi < \gamma} A_\xi = \emptyset$.
(b) $\bigcap_{\xi < \gamma} A_\xi \neq \emptyset$, and $\bigcap_{\xi < \gamma} A_\xi \not\subseteq A_\xi, \forall \xi < \gamma$.

**Proof.** We divide the proof into several steps. Since $\mathcal{F}$ fails to have the $\alpha$-i.p., there is a cardinal $\theta < \alpha$ and a family $(A_\eta)_{\eta < \theta}$ with $\bigcap A_\eta = \emptyset$. Moreover we can suppose, without loss of generality, that $\theta$ is the smallest cardinal having the above property, i.e. $\bigcap_{\eta < \theta} A_\eta \neq \emptyset, \forall \theta' < \theta$.

(I) $\forall \eta < \theta \exists \tilde{\eta} < \theta / \bigcap_{\eta < \tilde{\eta}} A_\eta \not\subseteq A_{\tilde{\eta}}$.

For, take $y \in \bigcap_{\eta < \tilde{\eta}} A_\eta$ and $\lambda < \theta$ such that $y \notin A_\lambda$. Then $\bigcap_{\eta < \tilde{\eta}} A_\eta \not\subseteq A_\lambda$. If $\tilde{\eta}$ is the smallest ordinal $\mu$ such that $\bigcap_{\eta < \tilde{\eta}} A_\eta \not\subseteq A_\mu$, we have: $\bigcap_{\eta < \tilde{\eta}} A_\eta \not\subseteq A_\lambda$ and $\bigcap_{\eta < \tilde{\eta}} A_\eta \subseteq \bigcap_{\eta < \tilde{\eta}} A_\eta$.

(II) $\tilde{\eta} > \eta$. Obvious.

(III) $\tilde{\eta} = \eta$. Let us suppose $\tilde{\eta} > \eta$. Then, by (I) and (II) we have the following chain of inclusions: $\bigcap_{\eta < \eta} A_\eta \subseteq \bigcap_{\eta < \eta} A_\eta \subseteq \bigcap_{\eta < \tilde{\eta}} A_\eta \subseteq A_\eta$ which is absurd.

Let us denote by $K$ the set of all $\eta$ for $\eta < \theta$.

(IV) $\bigcap_{\eta \in K} A_\eta = \emptyset$. For $z \in \bigcap_{\eta \in K} A_\eta$, let us call $\eta_0$ the first index with $z \notin A_{\eta_0}$. It is easy to verify that $z \notin A_{\eta_0}$.

(V) $\bigcap_{\eta < \eta_0} A_\eta \not\subseteq A_{\eta_0}$. This follows from (III) and from the fact that $\bigcap_{\eta < \eta} A_\eta \subseteq \bigcap_{\eta < \tilde{\eta}} A_\eta$.

Now we set $|K| = \gamma$. Of course $\gamma < \theta < \alpha$, and indexing $K$ by ordinals $\xi < \gamma$ we obtain respectively (a) from (IV) and (b) from (V).

3. **Main result.** We can now prove our main result. We have the following

**Theorem.** Let $\beta$ be a cardinal number. $\beta$ is Ulam-stable if and only if the space $\nu D(\beta)$ is paracompact.

**Proof.** If $\beta$ is nonmeasurable, the theorem is trivially true; hence we assume $\beta > m$.

**Necessity.** Let us suppose $\beta$ Ulam-stable, and show that $\nu D(\beta)$ is paracompact. Let $\mathcal{F}$ be a filter on $D(\beta)$ such that $f(\mathcal{F})$ has a cluster point, $\forall f \in \mathcal{F}_{D(\beta)}$. It is enough to prove that $\mathcal{F}$ has the m.i.p.; then by the hypothesis $\mathcal{F}$ is contained in an ultrafilter $A_p, p \in \nu D(\beta)$, and consequently has a cluster point in $\nu D(\beta)$, which turns out to be paracompact for Lemma 1. Suppose on the contrary that $\mathcal{F}$ has not the m.i.p. Then, by Lemma 2, there is a nonmeasurable cardinal $\gamma$ and a family $(A_\xi)_{\xi < \gamma}, A_\xi \in \mathcal{F}$, for which the properties (a), (b) of the quoted lemma hold.

For every $\xi < \gamma$, let $x_\xi$ be a point such that $x_\xi \in \bigcap_{\xi < \xi} A_\xi$ and $x_\xi \notin A_\xi$. Let us pose $D = (x_\xi)_{\xi < \gamma}$. Now we are going to define a map $\lambda: D(\beta) \rightarrow D$ in the following way: $\lambda(x) = x_\xi$, where $\xi$ is the first ordinal with $x \notin A_\xi$.

Since $D$ is a discrete space of nonmeasurable cardinality, it is realcompact. This implies that $\lambda$ is continuously extendable to $\nu D(\beta)$, namely $\lambda \in \mathcal{F}_{D(\beta)}$. 

On the other hand, the filter $\lambda(F)$ does not have cluster points in $D$, because the family $(\lambda(A_\mu)_{\mu<\gamma})$ is free. For it, if $z \in \bigcap_{\mu<\gamma} \lambda(A_\mu)$ then $z = x_\mu$ for a certain index $\mu < \gamma$ and moreover there is a point $x$ in $A_\mu$ with $z = \lambda(x)$. But by the definition of $\lambda$, $\lambda(x) = x_\mu$ where $x \not\in A_\mu$, so $x_\mu$ cannot belong to $\lambda(A_\mu)$. This is absurd, since we had supposed that $f(S)$ had a cluster point, for each $f \in F$, consequently $F$ must have the m.i.p., and the proof is completed.

**Sufficiency.** Let us suppose $\nu D(\beta)$ paracompact, and let $F$ be a filter on $D(\beta)$ with m.i.p. We must show that $F$ is contained in an ultrafilter $A_p$, for $p \in \nu D(\beta)$. By Lemma 1, it is enough to prove that $f(S)$ clusters, for each $f \in F_{D(\beta)}$. Let us point out firstly that $f(D(\beta))$ must be a realcompact metric space $\forall f \in F_{D(\beta)}$. Indeed, if $f(D(\beta))$ contained a discrete, closed and measurable set, $f$ could not be extended to $\nu D(\beta)$. The proof is easy and we omit it.

Suppose now that there is a $g \in F_{D(\beta)}$ such that $g(F)$ does not cluster in a metric, realcompact space $D$. Since $g(F)$ has not cluster points there exists a locally finite open cover $(U_i)_{i<\sigma}$, $U_i \subseteq D$, and a family $(F_i)_{i<\sigma}$, $F_i \in F$, with $U_i \cap g(F_i) = \emptyset$.

Clearly $\sigma$ is not measurable, because if it were, one could select a discrete, closed, measurable set contrary to the realcompactness of $D$. We have $\bigcap_{i}(D - U_i) = \emptyset$ with $(D - U_i) \in g(F)$, against the fact that $F$ and consequently $g(F)$ have the m.i.p. This completes the proof of our theorem.

**References**


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