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A TOPOLOGICAL CHARACTERIZATION OF A CLASS OF CARDINALS

RODOLFO TALAMO

ABSTRACT. Let $\mathfrak{m}$ be the first measurable cardinal. We say that a cardinal α is Ulam-stable if, on the discrete space $D(\alpha)$ of cardinal α, every filter with $\mathfrak{m}$-intersection property can be extended to an ultrafilter with $\mathfrak{m}$-intersection property. The main result we prove is the following: α is Ulam-stable if and only if its Hewitt-Nachbin realcompletion $\upsilon D(\alpha)$ is paracompact.

0. Introduction. Until now, various classes of cardinals have been considered in mathematical literature (see [1], [6]); a certain amount of attention has been devoted to characterize, in topological terms, some set-theoretic relations connected with them.

Keisler and Tarski [6] introduced a binary relation $\mathfrak{R}$, in the class of all cardinals, which Comfort-Negrepontis in their paper [1] modified as follows: $\alpha \; \partial \; \beta$ provided that there is, on the discrete space of cardinal β, an α-complete filter that cannot extend to an α-complete ultrafilter. In the paper referred to, such set-theoretic relation is described from a topological point of view. The aim of the present paper goes along these lines: we find a topological characterization for the class of all cardinals which do not satisfy the above relation for fixed $\alpha = \mathfrak{m}$ (first Ulam-measurable cardinal). We call them Ulam-stable cardinals and denote their class by $\mathfrak{I}_{\mathfrak{m}}$. It is well known that if α is a nonmeasurable cardinal (in the sense of [5]) then the discrete space $D(\alpha)$ is realcompact, and thus it coincides with its Hewitt-Nachbin realcompletion $\upsilon D(\alpha)$, which is obviously paracompact.

Assuming the existence of an Ulam-measurable cardinal, the problem we solve here is to determine the class of all cardinals ξ for which $\upsilon D(\xi)$ is paracompact. We prove that $\upsilon D(\xi)$ is paracompact if and only if ξ is Ulam-stable, i.e., ξ belongs to $\mathfrak{I}_{\mathfrak{m}}$.

1. Basic notions and results. We denote ordinal numbers by Greek letters; a cardinal number is an initial ordinal. We indicate by υ the first infinite cardinal and by α^+ the smallest cardinal greater than α. A cardinal α is said to be *regular* if it cannot be expressed as the sum of fewer than α cardinals each smaller than α. The discrete space of cardinal β is denoted by $D(\beta)$. A filter $\mathfrak{F}$ on $D(\beta)$ has the *α-intersection property* (abbr. α.i.p.) if $\bigcap \mathfrak{B} \neq \emptyset$ whenever $\mathfrak{B} \subseteq \mathfrak{F}$ and $|\mathfrak{B}| < \alpha$; moreover, if $\bigcap \mathfrak{B} \in \mathfrak{F}$ for $\mathfrak{B} \subseteq \mathfrak{F}$ and $|\mathfrak{B}| < \alpha$, then $\mathfrak{F}$ is said to be *α-complete*. An ultrafilter with α.i.p. is clearly α-complete. We recall here that a cardinal α is

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(*Ulam*)-*measurable* if there exists a nonprincipal υ^+-complete ultrafilter on $D(\alpha)$; let $\mathfrak{m}$ denote the first measurable cardinal. It is known that if $\alpha > \mathfrak{m}$ then α is a measurable cardinal.

Furthermore, $D(\alpha)$ is not realcompact and its Hewitt-Nachbin realcompletion $\upsilon D(\alpha)$ can be thought of as the space of all m-complete ultrafilters on $D(\alpha)$. Other equivalent definitions for measurable cardinals (involving measures) are to be found in [3] and [5]. We say that a cardinal α is $\mathfrak{m}$-*stable* (or *Ulam-stable*) if, on $D(\alpha)$, every filter with $\mathfrak{m}$.i.p. extends to an $\mathfrak{m}$-complete ultrafilter. We say that α is *strongly measurable* if α is regular and every filter on $D(\alpha)$ having the α.i.p. extends to an α-complete ultrafilter. We observe that if α is measurable, every $\mathfrak{m}$-complete filter on $D(\alpha)$ extends to an $\mathfrak{m}$-complete ultrafilter if and only if every filter with $\mathfrak{m}$.i.p. does.

Furthermore, we need some topological notions and results. Let X be a Tychonoff space; M a dense subset of X.

We denote by $\mathfrak{F}_{\mathfrak{M}}$ the class of all mappings of M, taking values in a metric space, which can be continuously extended to X. If $\mathfrak{F}$ is a flter on M, we denote by $\bar{\mathfrak{F}}$ the extension of $\mathfrak{F}$ to X, namely the filter on X generated by $\mathfrak{F}$. We state here without proof two theorems.

THEOREM A (CORSON [4]). *For a Tychonoff space X the following are equivalent*:

(I) *X is paracompact*;

(II) *If $\mathfrak{F}$ is a filter on X such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ has a cluster point in X.*

The second result we quote here was obtained jointly by W. W. Comfort and Teklehaimanot Retta and communicated to me February 1, 1977.

THEOREM B (COMFORT-RETTA). *If $\upsilon D(m)$ is paracompact, then m is strongly measurable.*

COROLLARY. *Suppose $\alpha \geqslant \mathfrak{m}$. If $\upsilon D(\alpha)$ is paracompact, then $\mathfrak{m}$ is strongly measurable.*

2. Lemmas. We need two lemmas. The first is a strengthened version of Theorem A.

LEMMA 1. *Let X be a Tychonoff space and M a dense subset of X. Then the following are equivalent*:

(I) *X is paracompact*;

(II) *If $\mathfrak{F}$ is a filter on M such that $f(\mathfrak{F})$ has a cluster point for each $f \in \mathfrak{F}_{\mathfrak{M}}$, then the extension $\bar{\mathfrak{F}}$ of $\mathfrak{F}$ to X has a cluster point in X.*

PROOF. (I) $\rightarrow$ (II) is obvious. For the converse, let $\mathfrak{F}$ be a filter on X such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped: we must show that $\mathfrak{F}$ clusters in X. Take the n.b.d. filter $\mathfrak{U}(\mathfrak{F})$ and its trace $\mathfrak{G}$ on M. It can be easily checked that $f(\mathfrak{G})$ has a cluster point for each $f \in \mathfrak{F}_{\mathfrak{M}}$ and thus $\bar{\mathfrak{G}}$ clusters in X by the hypothesis.

Consequently, $\mathfrak{F}$ has a cluster point in X and so X is paracompact applying Theorem A.

LEMMA 2. *Let α, β be infinite cardinals with $\beta \geqslant \alpha$ and $\mathfrak{F}$ a filter on $D(\beta)$ which fails to have the α.i.p. Then there is a cardinal $\gamma < \alpha$ and a family $(A_\xi)_{\xi<\gamma}$, $A_\xi \in \mathfrak{F}$ with the following properties*:

(a) $\bigcap_{\xi<\gamma} A_\xi = \varnothing$.

(b) $\bigcap_{\xi'<\xi} A_{\xi'} \neq \varnothing$, *and* $\bigcap_{\xi'<\xi} A_{\xi'} \nsubseteq A_\xi$, $\forall \xi < \gamma$.

PROOF. We divide the proof into several steps. Since $\mathfrak{F}$ fails to have the α.i.p., there is a cardinal $\theta < \alpha$ and a family $(A_\eta)_{\eta<\theta}$ with $\bigcap A_\eta = \varnothing$. Moreover we can suppose, without loss of generality, that θ is the smallest cardinal having the above property, i.e. $\bigcap_{\eta<\theta'} A_\eta \neq \varnothing, \forall \theta' < \theta$.

(I) $\forall \eta < \theta\ \exists \bar{\eta} < \theta / \bigcap_{\eta'<\eta} A_{\eta'} \nsubseteq A_{\bar{\eta}}$.

For, take $y \in \bigcap_{\eta'<\eta} A_{\eta'}$ and $\lambda < \theta$ such that $y \notin A_\lambda$. Then $\bigcap_{\eta'<\eta} A_{\eta'} \nsubseteq A_\lambda$. If $\bar{\eta}$ is the smallest ordinal μ such that $\bigcap_{\eta'<\eta} A_{\eta'} \nsubseteq A_\mu$, we have: $\bigcap_{\eta'<\eta} A_{\eta'} \nsubseteq A_{\bar{\eta}}$ and $\bigcap_{\eta'<\eta} A_{\eta'} \subseteq \bigcap_{\eta'<\bar{\eta}} A_{\eta'}$.

(II) $\bar{\eta} \geqslant \eta$. Obvious.

(III) $\bar{\bar{\eta}} = \bar{\eta}$. Let us suppose $\bar{\bar{\eta}} > \bar{\eta}$. Then, by (I) and (II) we have the following chain of inclusions: $\bigcap_{\eta'<\eta} A_{\eta'} \subseteq \bigcap_{\eta'<\bar{\eta}} A_{\eta'} \subseteq \bigcap_{\eta'<\bar{\bar{\eta}}} A_{\eta'} \subseteq A_{\bar{\eta}}$ which is absurd.

Let us denote by K the set of all $\bar{\eta}$ for $\eta < \theta$.

(IV) $\bigcap_{\bar{\eta}\in K} A_{\bar{\eta}} = \varnothing$. For $z \in \bigcap_{\bar{\eta}\in K} A_{\bar{\eta}}$, let us call η_0 the first index with $z \notin A_{\eta_0}$. It is easy to verify that $z \notin A_{\bar{\eta}_0}$.

(V) $\bigcap_{\bar{\eta}'<\bar{\eta}} A_{\bar{\eta}'} \nsubseteq A_{\bar{\eta}}$. This follows from (III) and from the fact that $\bigcap_{\eta'<\bar{\eta}} A_{\eta'} \subseteq \bigcap_{\bar{\eta}'<\bar{\eta}} A_{\bar{\eta}'}$.

Now we set $|K| = \gamma$. Of course $\gamma \leqslant \theta < \alpha$, and indexing K by ordinals $\xi < \gamma$ we obtain respectively (a) from (IV) and (b) from (V).

3. Main result. We can now prove our main result. We have the following

THEOREM. *Let β be a cardinal number. β is Ulam-stable if and only if the space $\upsilon D(\beta)$ is paracompact.*

PROOF. If β is nonmeasurable, the theorem is trivially true; hence we assume $\beta \geqslant \mathfrak{m}$.

Necessity. Let us suppose β Ulam-stable, and show that $\upsilon D(\beta)$ is paracompact. Let $\mathfrak{F}$ be a filter on $D(\beta)$ such that $f(\mathfrak{F})$ has a cluster point, $\forall f \in \mathfrak{F}_{D(\beta)}$. It is enough to prove that $\mathfrak{F}$ has the $\mathfrak{m}$.i.p; then by the hypothesis $\mathfrak{F}$ is contained in an ultrafilter A_p, $p \in \upsilon D(\beta)$, and consequently has a cluster point in $\upsilon D(\beta)$, which turns out to be paracompact for Lemma 1. Suppose on the contrary that $\mathfrak{F}$ has not the $\mathfrak{m}$.i.p. Then, by Lemma 2, there is a nonmeasurable cardinal γ and a family $(A_\xi)_{\xi<\gamma}$, $A_\xi \in \mathfrak{F}$, for which the properties (a), (b) of the quoted lemma hold.

For every $\xi < \gamma$, let x_ξ be a point such that $x_\xi \in \bigcap_{\xi'<\xi} A_{\xi'}$ and $x_\xi \notin A_\xi$. Let us pose $D = (x_\xi)_{\xi<\gamma}$. Now we are going to define a map λ: $D(\beta) \to D$ in the following way: $\lambda(x) = x_\xi$, where ξ is the first ordinal with $x \notin A_\xi$.

Since D is a discrete space of nonmeasurable cardinality, it is realcompact. This implies that λ is continuously extendable to $\upsilon D(\beta)$, namely $\lambda \in \mathfrak{F}_{D(\beta)}$.

On the other hand, the filter $\lambda(\mathfrak{F})$ does not have cluster points in D, because the family $(\lambda(A_\xi)_{\xi<\gamma})$ is free. For it, if $z \in \bigcap_{\xi<\gamma} \lambda(A_\xi)$ then $z = x_\mu$ for a certain index $\mu < \gamma$ and moreover there is a point x in A_μ with $z = \lambda(x)$. But by the definition of λ, $\lambda(x) = x_\rho$ where $x \notin A_\rho$, so x_μ cannot belong to $\lambda(A_\mu)$. This is absurd, since we had supposed that $f(\mathfrak{F})$ had a cluster point, for each $f \in \mathfrak{F}_{D(\beta)}$; consequently $\mathfrak{F}$ must have the m.i.p., and the proof is completed.

Sufficiency. Let us suppose $vD(\beta)$ paracompact, and let $\mathfrak{F}$ be a filter on $D(\beta)$ with m.i.p. We must show that $\mathfrak{F}$ is contained in an ultrafilter A_p, for $p \in vD(\beta)$. By Lemma 1, it is enough to prove that $f(\mathfrak{F})$ clusters, for each $f \in \mathfrak{F}_{D(\beta)}$. Let us point out firstly that $f(D(\beta))$ must be a realcompact metric space $\forall f \in \mathfrak{F}_{D(\beta)}$. Indeed, if $f(D(\beta))$ contained a discrete, closed and measurable set, f could not be extended to $vD(\beta)$. The proof is easy and we omit it.

Suppose now that there is a $g \in \mathfrak{F}_{D(\beta)}$ such that $g(\mathfrak{F})$ does not cluster in a metric, realcompact space D. Since $g(\mathfrak{F})$ has not cluster points there exists a locally finite open cover $(U_i)_{i<\sigma}$, $U_i \subseteq D$, and a family $(F_i)_{i<\sigma}$, $F_i \in \mathfrak{F}$, with $\overline{U_i} \cap \overline{g(F_i)} = \varnothing$.

Clearly σ is not measurable, because if it were, one could select a discrete, closed, measurable set contrary to the realcompactness of D. We have $\bigcap_i (D - U_i) = \varnothing$ with $(D - U_i) \in g(\mathfrak{F})$, against the fact that $\mathfrak{F}$ and consequently $g(\mathfrak{F})$ have the m.i.p. This completes the proof of our theorem.

References

1. W. W. Comfort and S. Negrepontis, *Some topological properties associated with measurable cardinals*, Fund. Math. **69** (1970), 191–205.

2. ______, *The theory of ultrafilters*, Springer-Verlag, New York, 1974.

3. ______, *Continuous pseudometrics*, Dekker, New York, 1975.

4. H. H. Corson, *The determination of paracompactness by uniformities*, Amer. J. Math. **80** (1958), 185–190.

5. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

6. H. J. Keisler and A. Tarsky, *From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones*, Fund. Math. **53** (1964), 225–308.

Istituto di Geometria dell'Università di Torino, Via P. Amedeo, 8-10123 Torino, Italy