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$\beta(X)$ CAN BE FRÉCHET

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ABSTRACT. A class of spaces is defined which share many properties of Gillman and Jerison's space ψ. These spaces are used to generalize a theorem of Malykhin, showing that certain one point compactifications are Stone-Čech compactifications. This is used to construct a space whose Stone-Čech compactification is a Fréchet space (under a set theoretic assumption which follows, for example, from the continuum hypothesis).

1. Introduction. It is well known that a first countable compact space cannot be the Stone-Čech compactification of a proper subspace [**1**, 9.7, p. 132]. Malykhin, however, proved that a proper Stone-Čech compactification can be sequential [**2**, Corollary 1, p. 19]. The heart of the proof was a theorem that certain products involving the one point compactification of Gillman and Jerison's space ψ [**1**, 5I, p. 79] are proper Stone-Čech compactifications. In this paper we show that a class of spaces which are like ψ itself in many respects can be substituted for ψ in Malykhin's theorem.

A space is a Fréchet space if, whenever a point p is a limit point of a set, there is a sequence from the set converging to p. While Malykhin's theorem by itself cannot be used to construct a proper Stone-Čech compactification which is Fréchet, the generalization proved in this paper does give rise to such a space. The construction makes use of a set theoretic axiom which follows, for example, from the continuum hypothesis. I do not know if a similar construction can be carried out in ZFC.

2. ψ-like spaces. A ψ-like space is a topological space built in the following way. It consists of a base space B and a discrete set of limit points D having these features. The base space is a disjoint union of infinitely many compact 0-dimensional Hausdorff spaces. These spaces are called the levels of the base space (in ψ itself, each level is a singleton, and there are countably many levels). There is an infinite collection C of sequences $(O(i))$ of open subsets of the base space. Each $O(i)$ is to be a clopen subset of some level and if i is different from j, then $O(i)$ and $O(j)$ come from different levels. The sequences are to be almost disjoint in this sense: if $(O(i))$ and $(V(j))$ are two different sequences in the collection, then $O(i) \cap V(j)$ must be empty (which can happen if $O(i)$ and $V(j)$ come from different levels, or if they are disjoint clopen subsets of the same level) for all but finitely many pairs of indices (i, j). Finally, C is to be maximal with respect to these

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properties. This implies that if $(W(i))$ is any sequence of open sets with each $W(i)$ coming from a different level, then there is a sequence $(O(i))$ in C such that $O(i) \cap W(j) \neq \varnothing$ for infinitely many pairs (i, j). Notice that this in turn implies that there are subsequences $(W(i_k))$ and $(O(j_k))$ such that $W(i_k) \cap O(j_k)$ is not empty for all k. The set D of limit points is disjoint from B and contains exactly one point for each sequence in the defining set of sequences C. D is indexed as $\{l(s): s \in C\}$. If $s = (O(i)) \in C$, a basic neighborhood of $l(s)$ is $\{l(s)\} \cup \bigcup O(i)$ $(i \geqslant n)$, where $n \in \omega$. Thus $l(s)$ has a countable basis. We define the complete neighborhood of $l(s)$ to be $\{l(s)\} \cup \bigcup O(i)$ $(i \in \omega)$.

Any ψ-like space shares the following properties of ψ:

(i) It is Hausdorff (the almost disjoint condition on sequences in C guarantees this).

(ii) It is locally compact.

(iii) It is pseudocompact.

(iv) It has no dense countably compact subset.

(v) If an open subset of the base space has infinitely many limit points in D, then it has uncountably many.

Furthermore, if the base space is first countable then so is the ψ-like space, and if the base space is sequential, then the one point compactification of the ψ-like space is sequential. Unlike for ψ itself, we cannot rule out the possibility that the one point compactification of a ψ-like space is Fréchet. It is instructive to see why the one point compactification of ψ is not Fréchet. In any ψ-like space X, the base space is dense and a sequence of open sets in the base space has a limit point in X, either by virtue of the maximality of C or the fact that levels are compact (this, of course, is why ψ-like spaces are pseudocompact). In ψ itself, though, since the levels are singletons, this says that any sequence of points in the base space B has a limit point in ψ i.e. that B is a dense conditionally compact subset of ψ, where we say that a subset S of a space is conditionally compact if every infinite subset of S has a limit point in the space. Now in the one point compactification of a ψ-like space, the added ideal point is a limit point of the base space, but since the base space in ψ is conditionally compact, no sequence from the base space converges to the ideal point. Thus the one point compactification of ψ is not Fréchet.

3. Generalizing Malykhin's theorem.

DEFINITION. A point p in a topological space is a Fréchet point if whenever p is a limit point of a set, then there is a sequence from the set converging to p. Thus, a space is Fréchet iff every point is a Fréchet point.

THEOREM 3.1. *Suppose $X = B \cup D$ is a ψ-like space with base space B and set of limit points D, and let $X \cup \{\alpha\}$ be the one point compactification of X. Suppose Y is a compact space and $p \in Y$ is a nonisolated Fréchet point. Then $X \cup \{\alpha\} \times Y = \beta(X \cup \{\alpha\} \times Y - (\alpha, p))$.*

PROOF. We will show that if f: $X \cup \{\alpha\} \times Y - (\alpha, p) \to R$ is a bounded continuous function, then f can be extended to $X \cup \{\alpha\} \times Y$. Note: When we say a sequence $(p(i))$ converges to p in Y, we will imply that $p(i) \neq p$ for all i (recall that p is not isolated).

LEMMA. *Suppose $(p(i))$ $(i \in \omega)$ converges to p, $(q(i))$ $(i \in \omega)$ converges to p, and both the sequence $(f(\alpha, p(i)))$ and the sequence $(f(\alpha, q(i)))$ converge in R. Then those latter sequences converge to the same real number.*

PROOF OF LEMMA. Suppose $(f(\alpha, p(i)))$ converges to r and $(f(\alpha, q(i)))$ converges to s with $r \neq s$. Let U and V be open subsets of R with disjoint closures containing r and s respectively. For some k and each $i > k$, there is an open subset $W(i) \subset X \cup \{\alpha\}$ containing α such that

$$f(W(i) \times \{p(i)\}) \subset U \quad \text{and} \quad f(W(i) \times \{q(i)\}) \subset V.$$

Thus for each $i > k$ there is an open set $O(i)$, a subset of some level of the base space, with $O(i) \subset W(i)$ such that if $i \neq j$ then $O(i)$ and $O(j)$ are subsets of different levels. The sequence $(O(i))$ $(i > k)$ has a limit point $l \in D$. But then $f(l, p)$ must be in $\mathrm{cl}(U) \cap \mathrm{cl}(V)$, contradicting the choice of U and V.

COROLLARY. *There is a number $r \in R$ such that if $(p(i))$ converges to p then $(f(\alpha, p(i)))$ converges to r. (The assumption we made that f is bounded is needed here.)*

Note that since p is a nonisolated Fréchet point, there is a sequence $(p(i))$ converging to p. We can now define $f(\alpha, p)$ to be the number r of the corollary. We need to show that if I as an open interval containing r then there is an open set $V \subset X \cup \{\alpha\}$ containing α and an open set $W \subset Y$ containing p such that $f(V \times W) \subset \mathrm{cl}(I)$.

Let I_1, I_2, and I_3 be open intervals with $r \in I_3 \subset \mathrm{cl}(I_3) \subset I_2 \subset \mathrm{cl}(I_2) \subset I_1 \subset \mathrm{cl}(I_1) \subset I$.

Claim 1. There are at most finitely many points $l \in D$ such that $f(l, p) \notin \mathrm{cl}(I_2)$.

PROOF. Suppose not, and let $\{l(i)\colon i \in \omega\}$ be an infinite subset of D with $f(l(i), p) \notin \mathrm{cl}(I_2)$ for each $i \in \omega$. There would have to be a sequence $(p(i))$ converging to p such that, for each i, $f(l(i), p(i)) \notin \mathrm{cl}(I_2)$ but $f(\alpha, p(i)) \in I_3$. Since $f(l(i), p(i)) \notin \mathrm{cl}(I_2)$, there is an open subset $U(i)$ of the base space (intersecting infinitely many levels) such that $f(U(i) \times \{p(i)\}) \cap \mathrm{cl}(I_2) = \varnothing$, with $l(i) \in \mathrm{cl}(U(i))$. But then $\bigcup U(i)$ $(i \in \omega)$ has uncountably many limit points in D and, for each such limit point l, $f(l, p) \notin I_2$. However since $f(\alpha, p(i)) \in I_3$, there is an open set $O(i)$ containing α such that $f(O(i) \times \{p(i)\}) \subset I_3$. Since $\bigcap O(i)$ $(i \in \omega)$ contains all but countably many points of D, there must be a

$$l \in \mathrm{cl}(\bigcup U(i)\ (i \in \omega) \cap \bigcap O(i) (i \in \omega)).$$

So $f(l, p)$ must be in $\mathrm{cl}(I_3)$, thus in I_2, which is impossible.

Let V_1 be the open subset of $X \cup \{\alpha\}$ formed by removing from $X \cup \{\alpha\}$ the complete open set of each of the finitely many points $l \in D$ with $f(l, p) \notin \mathrm{cl}(I_2)$. Then $f(D \cap V_1 \times \{p\}) \subset \mathrm{cl}(I_2)$.

Claim 2. $\{b \in B \cap V_1 \colon f(b, p) \notin \mathrm{cl}(I_1)\}$ intersects only finitely many levels of B.

PROOF. If not, there would be open sets $O(i)$ $(i \in \omega)$ with $O(i) \subset V_1$ such that $f(O(i) \times \{p\}) \cap \mathrm{cl}(I_1) = \varnothing$ and, if $i \neq j$, then $O(i)$ and $O(j)$ are subsets of different levels of B. But then the sequence $(O(i))$ would have a limit point $l \in D \cap V_1$. Since $\mathrm{cl}(I_2) \subset I_1, f(l, p) \notin \mathrm{cl}(I_2)$, contradicting the choice of V_1.

Let V be the open subset of $X \cup \{\alpha\}$ formed by removing from V_1 the finitely many (compact) levels of B which intersect $\{b \in B \cap V_1: f(b, p) \notin \mathrm{cl}(I_1)\}$. Then $f(V \times \{p\}) \subset \mathrm{cl}(I_1) \subset I$. If for each open set W containing p there were a point $p(W) \in W$ and a point $q(W) \in V$ such that $f(q(W), p(W)) \notin \mathrm{cl}(I)$, then the set $\{p(W)$: W is open containing $p\}$ would contain a sequence $(p(W(i)))$ $(i \in \omega)$ which converges to p. For each i, there would be an open set $O(i) \subset V$ containing $q(W(i))$, such that $f(O(i) \times \{p(W(i))\})$ misses $\mathrm{cl}(I)$. This, however, is impossible, since the sequence $(O(i))$ has a limit point $x \in V - \{\alpha\}$, and $f(x, p) \in I$. Thus for some open set W containing p, $f(V \times W) \subset \mathrm{cl}(I)$, as required, so the extension of f to $X \cup \{\alpha\} \times Y$ is continuous.

4. A Fréchet compactification. What must be true for the one point compactification $B \cup D \cup \{\alpha\}$ of a ψ-like space $B \cup D$ to be Fréchet? First of all, B must be Fréchet, or, equivalently, each level of B must be Fréchet. Since each point of D has a countable base, that will guarantee that $B \cup D$ is Fréchet. What if α is a limit point of a set $S \subset B \cup D$? If S contains infinitely many points of D, then any such infinite set converges to α. If $\mathrm{cl}(S)$ contains only finitely many points of D, then, if α is to be a limit point of S there must be a sequence of points $(b(i))$ $(i \in \omega)$ of $S \cap B$ such that if $i \neq j$ then $b(i)$ and $b(j)$ come from different levels of B and $b(i)$ is not in the complete open set of any $l \in \mathrm{cl}(S) \cap D$. But then $(b(i))$ converges to α. So we need only concern ourselves with the case where $S \subset B$ but $\mathrm{cl}(S) \cap D$ is infinite. In that case there is a countable subset

$$S' = \{s(i, j): i, j \in \omega\} \subset S$$

such that different points of S' come from different levels of B and for each i, there is a point $l(i) \in D$ with $(s(i, j))$ $(j \in \omega)$ converging to $l(i)$ and $i \neq j$ implies $l(i) \neq l(j)$. Of course, α will be a limit point of any such S', so we must make sure each such set has a sequence converging to α, i.e. each such S' must not be conditionally compact in $B \cup D$.

DEFINITION. If B is a base space for a ψ-like space (i.e. a disjoint union of 0-dimensional spaces) and C is a set of sequences of clopen subsets of B, we will say that a countable set $T = \{t(i, j): i, j \in \omega\}$ is relevant with respect to sequences of C if:

(1) different points of T come from different levels of B,

(2) for each i there is a sequence $C(i) \in C$ such that $\{t(i, j): j \in \omega\}$ is contained in the union of the terms of $C(i)$ and if $i \neq j$, then $C(i) \neq C(j)$.

Then, to summarize, if $B \cup D$ is a ψ-like space with defining set of sequences C, the one point compactification of $B \cup D$ is Fréchet if and only if each level of B is Fréchet and each subset of B relevant with respect to sequences of C is not conditionally compact in $B \cup D$.

We now construct a ψ-like space with these properties. The following set theoretic assumption, which follows from Martin's axiom or trivially from the continuum hypothesis, is used:

Every infinite maximal collection of almost disjoint subsets of a countable set has cardinality $c = \exp(\omega)$.

First, let the base space B have countably many levels, all copies of a space X with these properties:

(1) X is compact, Hausdorff, 0-dimensional.

(2) X is first countable.

(3) Every subset of X of cardinality less than c is nowhere dense.

Such a space can be constructed as follows: take a compact, 0-dimensional first countable space with density c. The product of countably many copies of such a space will have the properties required of X.

For notational purposes, we let $B = X \times \omega$ where ω has the discrete topology. Then $X \times \{n\}$ is the nth level of B, and if $p \in X$ and $S \subset X$ then (p, n) and $S \times \{n\}$ are the copies of p and S on the nth level. The space X of necessity has weight and cardinality c, and therefore so does B.

We will construct the defining set C of sequences of clopen sets in B by induction.

First consider all countable subsets $T \subset B$ with the property that different points of T come from different levels of B. Index this collection of sets with ordinals less than c in such a manner that each such T is indexed cofinally often.

Next consider all sequences of clopen subsets of B with the property that different terms of the sequence are subsets of different levels of B. Index these sequences as $\{s(\alpha): \alpha < c\}$. (This can be a one-to-one indexing.) We will use the following notation: $s(\alpha) = (O(\alpha, i) \times \{n(\alpha, i)\})$ $(i \in \omega)$, where $O(\alpha, i)$ is a clopen subset of X and $n(\alpha, i) \in \omega$. Thus if $i \neq j$ then $n(\alpha, i) \neq n(\alpha, j)$. Choose the indexing such that, for $\alpha, \beta \in \omega$, $s(\alpha)$ and $s(\beta)$ are almost disjoint in the sense required for a ψ-like space, i.e. $O(\alpha, i) \times \{n(\alpha, i)\} \cap O(\beta, j) \times \{n(\beta, j)\} = \varnothing$ for all but finitely many pairs (i, j). This will guarantee that we end up with an infinite collection of defining sequences.

For each $\alpha < c$, we will define by induction a set $A(\alpha)$ of ordinals less than or equal to α, a set $J(\alpha) \subset X$ with $\mathrm{card}(J(\alpha)) \leq \omega \cdot \mathrm{card}(\alpha)$ and a collection $C(\alpha)$ of sequences of clopen subsets of B. Also for some ordinals $\alpha < c$, we will define a sequence $s'(\alpha)$ of clopen subsets of B. In fact, we will define such a sequence iff $\alpha \in A(\alpha)$, and $C(\alpha)$ will be $\{s'(\beta): \beta \in A(\alpha)\}$. Finally for some ordinals $\alpha < c$, we will mark the set $T(\alpha)$ with a countably infinite subset $T'(\alpha) \subset T(\alpha)$. The construction of all this is as follows:

Suppose for all $\beta < \alpha$ we have defined $J(\beta)$, $A(\beta)$ and $C(\beta)$. Let $J'(\alpha) = \bigcup J(\beta)$ $(\beta < \alpha)$ (note that $\mathrm{card}(J'(\alpha)) \leq \omega \cdot \mathrm{card}(\alpha) < c$), $A'(\alpha) = \bigcup A(\beta)$ $(\beta < \alpha)$ and $C'(\alpha) = \bigcup C(\beta)$ $(\beta < \alpha)$. If $s(\alpha)$ is almost disjoint (in the sense required for a ψ-like space) from all the sequences in $C'(\alpha)$, then we define $s'(\alpha)$ as follows: For each $i \in \omega$, choose a nonempty clopen set $O'(\alpha, i) \subset O(\alpha, i) - J'(\alpha)$ and let $s'(\alpha) = (O'(\alpha, i) \times \{n(\alpha, i)\})$ $(i \in \omega)$. Also define $A(\alpha) = A'(\alpha) \cup \{\alpha\}$ and $C(\alpha) = C'(\alpha) \cup \{s'(\alpha)\}$. If $s(\alpha)$ is not almost disjoint from some sequence in $C'(\alpha)$, define $A(\alpha) = A'(\alpha)$, $C(\alpha) = C'(\alpha)$ and do not define $s'(\alpha)$. Next consider the set $T(\alpha)$. If $T(\alpha)$ is relevant with respect to the sequences in $C(\alpha)$, we define $T'(\alpha)$ as follows: Consider the set $T = \{T(\alpha) \cap \bigcup O'(\beta, i) \times \{n(\beta, i)\} \ (i \in \omega)$: $\beta \in A(\alpha)\}$. The infinite elements of T form an infinite collection of almost disjoint subsets of the countable set $T(\alpha)$ containing fewer than c sets. So, by the set theoretic assumption we are making, there is an infinite subset $T'(\alpha) \subset T(\alpha)$ which is almost disjoint from the sets in T. Mark $T(\alpha)$ with such a $T'(\alpha)$ and let $J(\alpha) = J'(\alpha) \cup \pi(T'(\alpha))$ where π: $B \to X$ is the projection onto X. Again, $\text{card}(J(\alpha)) \leq \omega \cdot \text{card}(\alpha)$. Otherwise, let $J(\alpha) = J'(\alpha)$.

Now let $C = \bigcup C(\alpha)$ $(\alpha < c)$. Sequences in C are pairwise almost disjoint in the sense required for a ψ-like space, and if $s(\alpha)$ is a sequence of clopen sets coming from different levels of B, $s(\alpha)$ is not almost disjoint from some sequence in $C(\alpha)$. Thus we can use C as the defining set of sequences for a ψ-like space. Give each $s \in C$ a convergent limit point $l(s)$ and let $D = \{l(s)\colon s \in C\}$. Then the ψ-like space $B \cup D$ has a Fréchet one point compactification, since, first of all, B is first countable. Secondly, if T is a countable subset of B relevant with respect to sequences in C, then T is relevant with respect to sequences in $C(\alpha)$ for some $\alpha < c$. But then $T = T(\beta)$ for some $\beta > \alpha$, and thus $T(\beta)$ was marked with a subset $T'(\beta)$. If $\gamma \in A(\beta)$, then $T'(\beta) \cap \bigcup O'(\gamma, i) \times \{n(\gamma, i)\}$ $(i \in \omega)$ is finite by choice of $T'(\beta)$, and if $\gamma > \beta$ and $s'(\gamma)$ is defined then $T'(\beta) \cap \bigcup O'(\gamma, i) \times \{n(\gamma, i)\}$ $(i \in \omega) = \varnothing$ by definition of $J(\beta)$. Thus $T'(\beta)$ has no limit point in $B \cup D$ so T is not conditionally compact.

5. $\beta(X)$ can be Fréchet. It is, of course, trivial for $\beta(X)$ to be Fréchet: X could be a compact Fréchet space itself! However, it is quite a different matter if X is not compact. First of all, if $\beta(X)$ is to be Fréchet, X must be pseudocompact, since otherwise $\beta(X)$ would contain a subspace homeomorphic to $\beta(N)$, no nonisolated (in the subspace topology) point of which is a Fréchet point. On the other hand, if $\beta(X) \neq X$, then for $\beta(X)$ to be Fréchet, X cannot be countably compact, or even have a dense conditionally compact subset, since, as the argument at the end of §2 shows, a noncompact space with a dense conditionally compact subset cannot have any Fréchet compactification.

If we apply Theorem 3.1 to the ψ-like space from §4, we get the following corollary.

COROLLARY 5.1. *If every infinite maximal collection of almost disjoint subsets of a countable set has cardinality c, then there is a noncompact space whose Stone-Čech compactification is Fréchet.*

PROOF. It follows from a theorem of Olson [3] that the product of a compact Fréchet space and a first countable space is Fréchet (in fact, countably bisequential). Thus if $B \cup D \cup \{\alpha\}$ is the one point compactification of the ψ-like space $B \cup D$ from §4, and I is the unit interval, then by Theorem 3.1,

$$\beta(B \cup D \cup \{\alpha\} \times I - (\alpha, 0)) = B \cup D \cup \{\alpha\} \times I,$$

which is Fréchet.

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