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SHORTER NOTES

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THE MAZUR-ORLICZ BOUNDED CONSISTENCY THEOREM

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ABSTRACT. An elementary and computationally easy proof of the Mazur-Orlicz bounded consistency theorem.

The theorem [3] of the title has been given many difficult proofs involving two-norm spaces and other functional analytic proofs of similar depth such as the Grothendieck interchange and completeness theorems. When elementary proofs were given they involved subtle and refined constructions and estimates. We have known an entirely elementary and computationally easy proof for a few years and have been motivated to publish it by the statement, [**4**, p. 542]: "Saks spaces . . . together with a special technique . . . leading to results which could not be obtained by previously known methods of functional analysis (e.g. to the Mazur-Orlicz theorem . . .)".

Standard notations and concepts as in [**7**], [**8**] are used as follows. We deal with *conservative* matrices i.e. A such that $c_A = \{x\colon Ax \in c\} \supset c$, the convergent sequences. Such A is *regular* if $\lim_A x = \lim x$ for $x \in c$ (where $\lim_A x = \lim Ax$), *multiplicative* 0 if $\lim_A x = 0$ for $x \in c$, *coregular* if $\chi(A) \neq 0$ where $\chi(A) = \lim_A e - \sum a_k$, ($e = \{1\}$, $a_k = \lim_n a_{nk}$), *conull* if $\chi(A) = 0$. It is well known that A is conull if and only if $e^{(n)} \to e$ weakly in the FK space c_A, where, for a sequence x, $x^{(n)}$ is the nth section $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ of x. See [**5**].

For an FK space E, let $W_E = \{x\colon x^{(n)} \to x \text{ weakly}\}$. For a matrix A let $W_A = W_{c_A}$. Now define $\Lambda_A(x) = \lim_A x - \sum a_k x_k$ for $x \in c_A \cap m$ (m = bounded sequences). Thus $\Lambda_A(e) = \chi(A)$.

LEMMA 1. *Let A be a conservative matrix. Then a bounded sequence $x \in W_A$ if and only if $\Lambda_A(x) = 0$.*

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If $x \in W_A$, $\sum_{k=1}^{n} a_k x_k = \lim_A x^{(n)} \to \lim_A x$ so $\Lambda_A(x) = 0$. Conversely let $\Lambda_A(x) = 0, f \in c'_A$. Then $f(x) = \alpha \lim_A x + \sum t_k x_k$ so

$$f(x^{(n)}) = \alpha \sum_{k=1}^{n} a_k x_k + \sum_{k=1}^{n} t_k x_k \to \alpha \sum a_k x_k + \sum t_k x_k = f(x).$$

Thus $x \in W_A$.

Lemma 2. *Let A be a conservative matrix. Then a bounded sequence $x \in W_A$ if and only if Ax is conull.*

By Ax we mean the matrix $(a_{nk} x_k)$. The result is immediate from Lemma 1 and the observation $\Lambda_A(x) = \chi(Ax)$.

Lemma 3. *Let A, B be conull matrices. Then $c_A \cap c_B$ contains a bounded divergent sequence.*

We may assume that A, B are multiplicative 0 by considering $(a_{nk} - a_k)$, $(b_{nk} - b_k)$. Form a matrix D by writing the rows of A, B alternately. Then D is multiplicative 0 and so, [1], D sums a bounded divergent sequence to 0.

Lemma 4. *Let A, B be conservative matrices with $c_A \cap m \subset c_B$. Then if A is conull, so is B.*

Let $D = B - \chi(B)I$. Then D is conull and by Lemma 3 there exists a bounded divergent sequence $x \in c_A \cap c_D$. By hypothesis $x \in c_B \cap c_D$. Thus $\chi(B)x = Bx - Dx$ is convergent, and so $\chi(B) = 0$.

Lemma 5. *Let A, B be conservative matrices with $c_A \cap m \subset c_B$. Then $W_A \cap m \subset W_B$.*

Let $x \in W_A \cap m$. By Lemma 2, Ax is conull. By Lemma 4, Bx is conull. By Lemma 2, $x \in W_B$.

The Bounded Consistency Theorem. *Let A, B be regular matrices with $c_A \cap m \subset c_B$. Then $\lim_B x = \lim_A x$ for $x \in c_A \cap m$.*

This is true for $x = e$. Thus we may assume $\Lambda_A(x) = 0$ since $\{x: \Lambda_A(x) = 0\}$ is of codimension one. But then $x \in W_A$ by Lemma 1. By Lemma 5, $x \in W_B$ so, by Lemma 1, $\Lambda_B(x) = 0$. Since A, B are regular $\Lambda_A = \lim_A$, $\Lambda_B = \lim_B$ and the theorem is proved.

Remark. Lemma 4 appeared in [6] and [2]. In the latter, it was deduced from the Bounded Consistency Theorem.

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