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# A NOTE ON THE MINIMUM CONDITION

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ABSTRACT. We characterize the minimum condition on subgroups in terms of generators.

It is well known that the maximum condition on subgroups of a group $G$ is equivalent to each subgroup of $G$ being finitely generated (for simplicity we discuss only groups, but the discussion applies equally well to rings, modules, etc.). For the minimum condition, however, no characterization in terms of generators is usually given. The purpose of this note is to put on record the existence of such a characterization.

DEFINITION. Let $S$ be an infinite subset of a group $G$. An element $s \in S$ is *essential in* $S$ if, given any infinite subset $T \subseteq S$, there exists an infinite subset $U \subseteq T$ such that the group $\langle U, s\rangle$ generated by $U$ and $s$ is different from the group $\langle U\rangle$ generated by $U$ alone.

THEOREM. *A group G satisfies the minimum condition on subgroups if and only if in any infinite subset of G there are only finitely many essential elements.*

PROOF. Let the infinite subset $S$ contain infinitely many essential elements, and let $T$ be the set of these elements. Let $t_1 \in T$. Then there exists an infinite subset $U_1$ of $T$ such that $\langle t_1, U_1\rangle = H_1 \neq H_2 = \langle U_1\rangle$. Let $t_2 \in U_1$. Then there exists an infinite subset $U_2$ of $U_1$ such that $\langle t_2, U_2\rangle \neq \langle U_2\rangle = H_3$, so certainly $H_3 \neq H_2$. Proceeding in the same way, we obviously get an infinite decreasing chain $H_1 \supset H_2 \supset H_3 \supset \cdots$.

Conversely, let $H_1 \supset H_2 \supset \cdots$ be an infinite properly decreasing chain of subgroups in $G$. Let each $i$ choose an element $s_i \in H_i - H_{i+1}$, and let $S = \{s_i\}$. Denote $S_j = \{s_i | i > j\}$. Then, given any infinite subset $T$ of $S$, $T \cap S_j$ is also infinite and $\langle T \cap S_j\rangle \neq \langle T \cap S_j, s_j\rangle$, because $T \cap S_j \subseteq H_{j+1}$. Thus all elements of $S$ are essential.

Note that the above proof shows that in the definition of essential elements we could take $T$ and $U$ to be cofinite in $S$.

The notion of essential elements could possibly be used to define other classes of groups, e.g., the condition that no infinite subset of $G$ contains essential elements if equivalent to all proper subgroups of $G$ by being finite.

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