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ISOMORPHISMS OF SUMS OF COUNTABLE BOOLEAN ALGEBRAS

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ABSTRACT. Denote by nA the sum of n copies of a Boolean algebra A. We prove that, for any countable Boolean algebra A, $nA \cong mA$ with $m > n$ implies $nA \cong (n + 1)A$.

In [2], W. Hanf constructed a Boolean algebra H isomorphic to the direct product $H \times H \times H$ but not to $H \times H$. This result of Hanf was strengthened by J. Ketonen in [4]: there exists even a countable Boolean algebra with this property.

In [1], P. R. Halmos examines various problems concerning isomorphism of products of Boolean algebras and asks the corresponding questions for sums in place of products. An analogy of the above result of W. Hanf is proved in [6]: there exists a Boolean algebra B isomorphic to $B + B + B$ but not to $B + B$. Even, every Boolean algebra is a homomorphic image of a Boolean algebra with this property. All the algebras with this property, constructed in [6], are very large. It is natural to ask whether there exists a countable Boolean algebra B isomorphic to $B + B + B$ but not to $B + B$. Let us notice that a weaker question about the existence of two nonisomorphic countable Boolean algebras A, B with $A + A$ isomorphic to $B + B$, was answered affirmatively in [5]. Nevertheless, there exists no countable Boolean algebra B isomorphic to $B + B + B$ but not to $B + B$. We prove here a stronger result, namely that, for countable Boolean algebras, $nB \cong mB$ with $m > n$ implies $nB \cong (n + 1)B$. This is stated in Theorem 3 of the present paper. Theorems 1 and 2 concern the Schroder-Bernstein property and the pseudo-indecomposability of countable Boolean algebras. They form steps in the proof of Theorem 3, but they could be interesting in themselves.

1. For Boolean algebras A, B, let us write $A < B$ iff $B \cong A \times C$ for some C.

DEFINITION. A Boolean algebra A is called *weakly pseudoindecomposable with respect to a Boolean algebra* A' if $A \cong B \times C$ implies either $A' < B$ or $A' < C$. A is called *weakly pseudoindecomposable* if it is weakly pseudoindecomposable with respect to itself.

2. THEOREM 1. *A weakly pseudoindecomposable countable Boolean algebra A has the Schroder-Bernstein property*: $A < B$, $B < A$ *implies* $A \cong B$.

PROOF. We prove the dual form of the theorem.

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(a) Let X be the Stone space of A. Since A is weakly pseudoindecomposable, X fulfils the following.

$$\left.\begin{array}{l}\text{If } X = Z_1 + Z_2, \text{ then } X \text{ is homeomorphic to a clopen}\\ \text{subset of either } Z_1 \text{ or } Z_2\end{array}\right\} \quad (1)$$

(where clopen is an abbreviation for closed-and-open). Let a Boolean algebra B with $A \lessdot B$ and $B \lessdot A$ be given; let Y be its Stone space. Then

$$\left.\begin{array}{l}X \text{ is homeomorphic to a clopen subset of } Y \text{ and}\\ Y \text{ is homeomorphic to a clopen subset of } X.\end{array}\right\} \quad (2)$$

We have to prove that X and Y are homeomorphic.

(b) We construct a decreasing sequence

$$X_1 \supset Y_1 \supset X_2 \supset Y_2 \supset X_3 \supset \cdots \quad (3)$$

of clopen subsets of $X_1 = X$ and homeomorphisms g_i of X_i onto X_{i+1} such that

(i) each X_i is homeomorphic to X, each Y_i is homeomorphic to Y,

(ii) $g_i(Y_i) = Y_{i+1}$,

(iii) $\bigcap_{i=1}^{\infty} X_i$ is a one-point set.

Then the mapping $\lambda: X_1 \to Y_1$ defined by

$$\lambda(x) = g_i(x) \quad \text{if } x \in X_i \setminus Y_i \text{ for some } i = 1, 2, \ldots,$$
$$\lambda(x) = x \quad \text{otherwise}$$

will be a one-to-one continuous mapping of $X = X_1$ onto $Y_1 \cong Y$, hence $X \cong Y$.

(c) We construct the sequence (3) which fulfils (i), (ii), (iii) as follows. Let $\mathfrak{B} = \{B_1, B_2, \ldots\}$ be a countable base of X consisting of nonvoid clopen subsets. Put $X_1 = X$, choose a clopen $Y_1 \subset X_1$ homeomorphic to Y and a clopen $Z_1 \subset Y_1$ homeomorphic to X (this is possible by (2)). By (1), there exists a clopen $X_2 \subset Z_1$, homeomorphic to X, such that either $X_2 \subset B_1$ or $X_2 \subset Z_1 \setminus B_1$. Choose a homeomorphism g_1 of X_1 into X_2. Now, we proceed by induction. If X_i, $i = 1, 2, \ldots, n$, and Y_j, g_j, $j = 1, 2, \ldots, n-1$, are already defined, put $Y_n = g_{n-1}(Y_{n-1})$, find a clopen $Z_n \subset Y_n$ homeomorphic to X and a clopen $X_{n+1} \subset Z_n$ homeomorphic to X such that either $X_{n+1} \subset B_n$ or $X_{n+1} \subset Z_n \setminus B_n$. Finally, choose a homeomorphism g_n of X_n onto X_{n+1}. The conditions (i) and (ii) follow immediately from the construction; (iii) is implied by the fact that $\mathfrak{B}$ is a base of X.

3. Remark. By Theorem 1, weakly pseudoindecomposable countable Boolean algebras are pseudoindecomposable in the sense of [3]: $A \cong B \times C$ implies either $A \cong B$ or $A \cong C$. The assumption that A is countable cannot be replaced by the weaker condition that A contains a countable dense subset. An example of a weakly pseudoindecomposable Boolean algebra with a countable dense subset, which is not pseudoindecomposable (and which fails to have the Schroder-Bernstein property) is the algebra $H \times 2$, where H is the Hanf's algebra isomorphic to $H \times 2 \times 2$ but not to $H \times 2$ (see [2])

4. If A is a countable Boolean algebra and $A \cong B \times F$, where F is nontrivial atomless (hence the free Boolean algebra on $\aleph_0$ generators), then $A + A \cong (B + B) \times F$. Thus, investigating isomorphisms of $A + \cdots + A$, one can restrict one-

self to the algebras A, which are not a product of a nontrivial atomistic algebra with the nontrivial atomless algebra F. Let us call them *essential*, for shortness. Thus, if A is essential and $A \cong B \times F$, then $A \cong B$. Clearly, if A is essential, then $A + \cdots + A$ is also essential.

THEOREM 2. *Let A be a countable Boolean algebra. If A is essential, then $A + A$ is weakly pseudoindecomposable with respect to A.*

Note. The converse is also true: if A is not essential, then $A + A$ is not weakly pseudoindecomposable with respect to A.

PROOF. We prove the dual form of the theorem. Let X be the Stone space of A. Then $X \times X$ is the Stone space of $A + A$. Since A is essential,

$$\text{either } \bar{S} = X \text{ or } \bar{S} \text{ is not open,} \tag{4}$$

where S denotes the set of all isolated points of X and $\bar{S}$ is the closure of S. Let $X \times X = V + W$. We have to prove that X is homeomorphic to a clopen subset of either V or W. For every $s \in S$, put

$$Z(s) = \{x \in X | (x, s) \in W\}, \qquad Z = \bigcap_{s \in S} Z(s).$$

Clearly, Z is a closed subset of X. For every $x \in X$ put

$$L(x) = \{t \in X | (x, t) \in W\}.$$

Since W is a clopen subset of $X \times X$, it has a form $\bigcup_{i=1}^{k} M_i \times N_i$, where M_i, N_i are clopen subsets of X. This implies that every $L(x)$ is a clopen subset of X and for every $x \in X$ there exists a clopen $U(x) \subset X$ such that $x \in U(x)$ and $L(y) = L(x)$ for all $y \in U(x)$. We have

$$x \in Z \Leftrightarrow L(x) \supset S,$$

hence Z is a clopen subset of X.

(α) Let us suppose that Z intersects $\bar{S}$. Since Z is open, it contains an element s of S. Since $L(s)$ is closed, it contains $\bar{S}$. Since $L(s)$ is a clopen set containing $\bar{S}$, it is homeomorphic to X by (4). Let λ be a homeomorphism of X onto $L(s)$. Since s is isolated in X, $\{s\} \times L(s)$ is clopen in $X \times X$. It is contained in W and f: $X \to X \times X$, sending x to $(s, \lambda(x))$, maps X onto it.

(β) Let us suppose that Z does not intersect $\bar{S}$. For every $s \in S$ put $Y(s) = Z(s) \setminus Z$. Then $Y(s)$ is a clopen subset of X and $\bigcap_{s \in S} Y(s) = \varnothing$. Since X is compact, there exists a finite set $\{s_1, \ldots, s_k\} \subset S$ such that $\bigcap_{j=1}^{k} Y(s_j) = \varnothing$. Define e: $X \setminus Z \to X \times X$ by $e(x) = (x, s_1)$ for $x \in (X \setminus Z) \setminus Y(s_1)$, $e(x) = (x, s_{i+1})$ for $x \in (\bigcap_{j=1}^{i} Y(s_j)) \setminus Y(s_{i+1})$, $i = 1, \ldots, k-1$. Then e is a homeomorphism of $X \setminus Z$ onto a clopen subset of $X \times X$ and $e(X \setminus Z) \subset V$. Since Z is a clopen subset of X and $Z \cap \bar{S} = \varnothing$, $X \setminus Z$ is homeomorphic to X by (4).

5. COROLLARY. *Let A be a countable essential Boolean algebra with $A + A \lessdot A$. Then A is pseudoindecomposable.*

Let us notice that if either $A + (A \times B) \cong A$ or $A + A + B \cong A$ for some B, then $A + A \lessdot A$.

6. For a Boolean algebra A, and a natural number n, let us denote by nA the sum of n copies of A.

THEOREM 3. *Let A be a countable Boolean algebra such that $nA \cong mA$ where $n < m$. Then $nA \cong (n+1)A$.*

PROOF. Let $nA \cong (n+k)A$ with $k > 1$. We have to prove that $nA \cong (n+1)A$. We may suppose that A is not atomless and that it is essential. We have $(n+1)A \leqslant (n+k)A \cong nA$ and, clearly, $nA \leqslant (n+1)A$. Consequently, it is sufficient to prove that nA has the Schroder-Bernstein property. By Theorem 1, it is sufficient to prove that nA is pseudoindecomposable. By §5, it is sufficient to prove that $nA + nA \leqslant nA$. Since $nA \cong (n+k)A \cong nA + kA$ with $k > 1$, we have $nA \cong (n + pk)A$ for all natural numbers p. Choose p such that $n + pk > 2n$. Then $nA + nA \cong 2nA \leqslant (n+pk)A \cong nA$.

7. REMARK. The conclusion of Theorem 3 is probably optimal. An example is given by R. S. Pierce [5] of a countable Boolean algebra A satisfying $A \ncong 2A \cong 3A$ and, as the referee announced to the author, there is an unpublished example of a countable A such that $A \ncong 2A \ncong 3A \cong 4A$, described by R. S. Pierce at the Summer AMS meeting in 1977.

8. OPEN PROBLEM. Which cardinal number is the smallest cardinality of a Boolean algebra A isomorphic to $A + A + A$ but not to $A + A$?

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