THE MEAN-VALUE ITERATION FOR SET-VALUED MAPPINGS

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Abstract. In this note Krasnoselskii's iteration procedure
\[ x_{n+1} = \frac{1}{2}(I + T)x_n \]
is extended to certain classes of set-valued mappings.

1. Introduction. Let \( C \) be a convex subset of a Banach space \( B \) and \( T \) a self-mapping of \( C \) and consider the following iteration process of a type introduced by Mann [7]: for an arbitrary starting point \( x_0 \in C \)
\[ x_{n+1} = (1 - c_n)x_n + c_nTx_n, \quad n = 0, 1, 2, \ldots \tag{*} \]
where \( c_n \in [a, b] \) for \( 0 < a < b < 1 \). The special case \( c_n = \frac{1}{2} \) for all \( n \) was first introduced by Krasnoselskii [5], who showed that the sequence converges to a fixed point of \( T \) if \( T \) is nonexpansive, \( B \) uniformly convex, and \( C \) compact. This result remains valid if \( c_n = \alpha \), \( 0 < \alpha < 1 \) (Schaefer [12]). Moreover, it is sufficient to assume that \( B \) is strictly convex (Edelstein [3]). Retaining uniform convexity, Browder and Petryshyn [1] assumed \( C \) to be closed and \( T \) demicompact. Under the latter conditions, the sequence \((*)\) converges to a fixed point of \( T \) if \( T \) is merely continuous and quasinonexpansive, that is, nonexpansive about its set of fixed points, assumed nonempty. (See Corollary which follows.) The iteration \((*)\) has been investigated by Senter and Dotson [13].

In this paper we shall consider an analogous iteration for a mapping \( T: C \rightarrow K(C) \), where \( K(C) \) is the family of nonempty compact subsets of \( C \). It is assumed that one fixed point \( z \) is known and that \( T \) is nonexpansive about this point, that is, for all \( x \in C \)
\[ D(Tx, Tz) < \|x - z\|, \]
where \( D \) is the Hausdorff metric on \( K(C) \). The iteration procedure is designed to generate additional fixed points.

Regarding the existence of fixed points, it was shown by Lim [6] that if \( C \) is a convex closed and bounded subset of a uniformly convex Banach space, then a nonexpansive mapping \( T: C \rightarrow K(C) \) has a fixed point. This result has recently been extended by Downing and Kirk [2].

2. The sequence. Let \( z \in Tz \) be the known fixed point. Since \( Tx \) is compact and \( D \) the Hausdorff metric, we can find for every \( x \in C \) a point \( p_x \in Tx \) such that
\[ \|z - p_x\| < D(Tz, Tx). \]
Using this information, suppose we construct a sequence \( \{x_n\} \) in \( C \) as follows: let \( x_0 \in C \) and \( p_0 \in Tx_0 \). Next let

\[
x_1 = (1 - c_0)x_0 + c_0p_0,
\]

where \( c_0 \in [a, b] \), \( 0 < a < b < 1 \). Then we can find \( p_1 \in Tx_1 \) such that

\[
\|z - p_1\| < D(Tz, Tx_1)
\]

by the prior comments. Now let

\[
x_2 = (1 - c_1)x_1 + c_1p_1.
\]

Since \( Tx_2 \) is compact, we can find \( p_2 \in Tx_2 \) such that

\[
\|z - p_2\| < D(Tz, Tx_2).
\]

Continuing in this manner

\[
x_{n+1} = (1 - c_n)x_n + c_np_n, \quad n = 0, 1, 2, \ldots,
\]

where \( c_n \in [a, b] \) for \( 0 < a < b < 1 \), \( p_n \in Tx_n \), and

\[
\|z - p_n\| < D(Tz, Tx_n).
\]

Since \( T \) is not even assumed to be quasinonexpansive, we do require continuity in the following sense.

**Definition 1.** A mapping \( T : C \to K(C) \) is **continuous** if for any sequence \( \{y_n\} \) in \( C \), \( y_n \to y \) implies that \( Ty_n \to Ty \).

**Definition 2** (Petryshyn [10]). A mapping \( U : C \to B \) of a subset \( C \) of a Banach space \( B \) into \( B \) is said to be **demicompact** if whenever \( \{x_n\} \subseteq C \) is a bounded sequence and \( \{x_n - Ux_n\} \) is a convergent sequence, then there exists a subsequence \( \{x_{n_k}\} \) which is convergent.

For set-valued mappings we have the following analogous definition.

**Definition 3.** A mapping \( T : C \to K(C) \) will be called **demicompact** if whenever \( \{x_n\} \subseteq C \) is a bounded sequence and \( \{\text{dist}(x_n, Tx_n)\} \) is a convergent sequence, then there is a subsequence \( \{x_{n_k}\} \) which is convergent.

In the proof of the first theorem we are going to need the following two lemmas.

**Lemma 1** (Schaefer [12]). Let \( B \) be a uniformly convex Banach space. Then for \( \varepsilon > 0 \), \( d > 0 \), and \( \alpha \in (0, 1) \) the inequalities

\[
\|x\| < d, \quad \|y\| < d, \quad \text{and} \quad \|x - y\| > \varepsilon
\]

imply that

\[
\|(1 - \alpha)x + \alpha y\| < \left[ 1 - 2\delta(\varepsilon/d) \right] \min(1 - \alpha, \alpha) d;
\]

\( \delta \) is strictly increasing.

**Lemma 2** (Nadler [9]). Let \( \{A_n\} \) be a sequence of sets in \( K(C) \) and suppose

\[
\lim_{n \to \infty} D(A_n, A_0) = 0, \quad \text{where} \quad A_0 \in K(C).
\]

Then if \( x_n \in A_n \), \( n = 1, 2, \ldots \), and if \( \lim_{n \to \infty} x_n = x_0 \), it follows that \( x_0 \in A_0 \).

3. Results.

**Theorem 1.** Let \( C \) be a nonempty convex closed subset of a uniformly convex Banach space \( B \). If \( T : C \to K(C) \) is a continuous demicompact mapping which is nonexpansive about a known fixed point \( z \), then for the sequence \( \{x_n\} \) defined previously, \( a \) there exists a subsequence \( \{x_{n_k}\} \) converging to a fixed point of \( T \) and
(b) every cluster point of \(\{x_n\}\) is a fixed point of \(T\). (In particular, every convergent subsequence of \(\{x_n\}\) converges to a fixed point.)

**Proof.** The first step is to show that for the sequence \(\{x_n\}\) constructed previously

\[
\|x_n - p_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

If not, then there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) and a number \(\varepsilon > 0\) such that

\[
\|x_{n_k} - p_{n_k}\| \geq \varepsilon. \tag{1}
\]

Since \(p_{n_k} \in Tx_{n_k}\),

\[
\|z - p_{n_k}\| < D(Tz, Tx_{n_k}) < \|z - x_{n_k}\|. \tag{2}
\]

Then by (1), (2) and Lemma 1 there exists \(\delta = \delta(\varepsilon/\|z - x_{n_k}\|) > 0\) such that

\[
\|z - x_{n_{k+1}}\| = \|z - (1 - c_{n_k})x_{n_k} - c_{n_k}p_{n_k}\|
= \|\|1 - c_{n_k}\|(z - x_{n_k}) + c_{n_k}(z - p_{n_k})\|
< (1 - \delta\gamma)\|z - x_{n_k}\|,
\]

where \(\gamma = 2 \min(1 - c_{n_k}, c_{n_k})\). From

\[
\|z - x_{n_{k+1}}\| = \|(1 - c_{n_k})(z - x_{n_k}) + c_{n_k}(z - p_{n_k})\| < \|z - x_{n_k}\|, \tag{3}
\]

the sequence \(\{\|z - x_n\|\}\) is nonincreasing, and since \(\delta\) is strictly increasing, the sequence

\[
\left\{ \delta(\varepsilon/\|z - x_{n_k}\|) \right\}
\]

in nondecreasing. Since we also have

\[
\|z - x_{n_k}\| < \|z - x_{n_{k+1}}\| < (1 - \delta\gamma)\|z - x_{n_{k-1}}\|
\]

for

\[
\delta = \delta(\varepsilon/\|z - x_{n_k}\|)
\]

and

\[
\gamma = 2 \min(1 - c_{n_{k-1}}, c_{n_{k-1}}),
\]

it follows that

\[
\|z - x_{n_k}\| \to 0 \quad \text{as} \quad j \to \infty.
\]

By (2) \(\|z - p_{n_k}\| \to 0\), whence \(\|x_{n_k} - p_{n_k}\| \to 0\) as \(j \to \infty\), contradicting statement (1).

Hence

\[
\|x_n - p_{n_k}\| \to 0 \quad \text{as} \quad n \to \infty, \tag{4}
\]

which was to be shown.

It now follows from (4) that \(\text{dist}(x_n, Tx_n) \to 0\) as \(n \to \infty\). Moreover, by (3), \(\{x_n\}\) is a bounded sequence. So by demicompactness there exists a subsequence \(\{x_{n_k}\}\) of
$\{x_n\}$ such that $x_n \to z_0 \in C$. Also, from
\[ \|p_n - z_0\| \leq \|p_n - x_n\| + \|x_n - z_0\|, \]
we have that $p_n \to z_0$. But $Tx_n \to Tz_0$ by continuity. Consequently, since $p_n \in Tx_n$, $z_0 \in Tz_0$ by Lemma 2.

Finally, if $w_0$ is a cluster point of $\{x_n\}$, there exists a subsequence converging to $w_0$, which is a fixed point by the above argument. This completes the proof.

Recall from §2 that $p_x \in Tx$ is a point for which $\|z - p_x\| < D(Tz, Tx)$. Suppose for every such $p_x \in Tx$ and $p_y \in Ty$, $T: C \to K(C)$ satisfies the condition
\[ D(Tx, Ty) < \alpha \|x - p_x\| + \beta \|y - p_y\| \quad (A) \]
for all $x, y \in C$ and $\alpha, \beta \in [0, \infty)$.

Then if $\alpha = \beta \in [0, \frac{1}{2}]$ and if $T$ is a point-to-point mapping, $T$ is a Kannan mapping, first introduced by Kannan [4]. Such mappings have been studied extensively.

If $p_w$ is chosen (without reference to the fixed point $z$) so that $\|w - p_w\| = \text{dist}(w, Tw)$ and if $\alpha = \beta = \frac{1}{2}$, then $T$ becomes a set-valued Kannan mapping. Such mappings were studied by Shiau, et al. [14], [15]. Clearly, any Kannan mapping is of Type A.

**Theorem 2.** If $T$ in Theorem 1 is of Type A, then $Tx_n \to Tz_0$, where $z_0 \in Tz_0$.

**Proof.** For every $\varepsilon > 0$ there exists $N > 0$ such that
\[ D(Tx_n, Tx_m) < \alpha \|x_n - p_n\| + \beta \|x_m - p_m\| < \varepsilon \]
for all $m, n > N$, so that $\{Tx_n\}$ is a Cauchy sequence. Since $C$ is complete, $(K(C), D)$ is complete (Michael [8]). Hence $Tx_n \to L \in K(C)$. Since $x_n \to z_0$, $Tx_n \to Tz_0$, so that $L = Tz_0$.

The result of Theorem 2 is, in one sense, the best possible. For if $T$ is a nonexpansive set-valued mapping, then the natural analogue of Krasnoselskii’s procedure is the following: let $x_0 \in C, q_0 \in Tx_0$, and $x_1 = \frac{1}{2}x_0 + \frac{1}{2}q_0$. Now choose $q_1 \in Tx_1$ such that
\[ \|q_1 - q_0\| < D(Tx_1, Tx_0). \]
In general, $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}q_n$, where $q_n \in Tx_n$ and
\[ \|q_n - q_{n-1}\| < D(Tx_n, Tx_{n-1}), \]
whence
\[ \|q_n - q_{n-1}\| < \|x_n - x_{n-1}\|. \]
This construction fails, however, as can be seen from the mapping $T: R \to K(R)$ defined by $Tx = [x - 1, x + 1]$. If for $Tx_n = [x_n - 1, x_n + 1]$, we choose $q_n = x_n + 1$, then the resulting sequence has no convergent subsequence.

If $T$ is a point-to-point mapping, then $q_n = Tx_n$, and $\{x_n\}$ reduces to Krasnoselskii’s iteration. Now suppose $T$ is continuous, demicompact, and quasinonexpansive with a nonempty set $F$ of fixed points. Returning to the sequence $(\ast)$, if $z \in F$, then
\[ \|z - Tx_n\| \leq \|z - x_n\|. \]
and, by the proof of Theorem 1, there exists a subsequence \( \{x_n\} \) for which 
\[ x_n \to z_0 = Tz_0. \]
Since \( \{\|z_0 - x_n\|\} \) is clearly nonincreasing, 
\[ x_n \to z_0. \]
This proves the following

**Corollary.** Let \( C \) be a nonempty convex closed subset of a uniformly convex Banach space. If \( T: C \to C \) is a continuous demicompact quasinonexpansive mapping with a nonempty set of fixed points, then the sequence \( (*) \) converges to a fixed point of \( T \).

This result is similar to Theorem 1.1' in [11] and Theorem 2 in [13].

**References**


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