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L^1-CONVERGENCE OF FOURIER SERIES WITH COEFFICIENTS OF BOUNDED VARIATION

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ABSTRACT. It is shown that a Telyakovskii [2] theorem and its generalization are special cases of a corollary to a theorem of Garrett and Stanojević [4]. Another theorem of Garrett and Stanojević is generalized for the case of coefficients of bounded variation of order $m > 1$.

1. The classical Kolmogorov [1] result was extended by Telyakovskii [2] to a class $\mathcal{S}$ of Fourier coefficients that contains the class of quasi-convex coefficients introduced by Kolmogorov [1]. The class $\mathcal{S}$ is defined in the following way.

DEFINITION 1.1. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{S}$ if there exists a monotone sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$, and $|\Delta a_k| < A_k$, for all k.

Let $\mathcal{BV}$ denote the class of null-sequences of bounded variation. Then

$$\{a_k\} \in \mathcal{S} \Rightarrow \{a_k\} \in \mathcal{BV},$$

for it follows from Definition 1.1 that the series $\sum_{k=1}^{\infty} |\Delta a_k|$ converges. Hence the point-wise limit f of

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

exists in $(0, \pi]$, whenever $\{a_k\} \in \mathcal{S}$. Since $\{a_k\} \in \mathcal{S}$ is equivalent to the Sidon [3] sufficient integrability condition, we have that if $\{a_k\} \in \mathcal{S}$ then the series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

is a Fourier series.

THEOREM A (TELYAKOVSKII [2]). *Let $\{a_k\} \in \mathcal{S}$. Then $f \in L^1(0, \pi)$, and*

$$\|s_n - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$a_n \lg n = o(1), \qquad n \to \infty.$$

However Theorem A is a special case of a Garrett and Stanojević [4] theorem. To state concisely that theorem we shall define a class of null-sequences.

DEFINITION 1.2. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{C}$ if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon)$, independent of n, and such that for all n,

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$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon,$$

where D_k is the Dirichlet kernel.

THEOREM B. *Let* $\{a_k\} \in \mathfrak{BV} \cap \mathcal{C}$. *Then* $f \in L^1(0, \pi)$, *and*

$$\|s_n - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$a_n \lg n = o(1), \qquad n \to \infty.$$

In fact Theorem B is a corollary to a more general theorem proved in [4]. More importantly, it has been proved in [4] that if $\{a_k\} \in \mathfrak{BV}$ and $a_n \lg n = o(1)$, $n \to \infty$, then Theorem A cannot be extended beyond Theorem B.

THEOREM C. *Let* $\{a_k\} \in \mathfrak{BV}$ *and let* $a_n \lg n = o(1)$, $n \to \infty$. *Then*

$$\|s_n - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$\{a_k\} \in \mathcal{C}.$$

To prove that $\mathcal{S} \subset \mathfrak{BV} \cap \mathcal{C}$ we shall give an equivalent definition of the class $\mathcal{S}$.

DEFINITION 1.3. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{S}^2$ if there exists a null-sequence $\{A_k\}$ of nonnegative numbers such that $\sum_{k=1}^{\infty} k|\Delta A_k| < \infty$ and $|\Delta a_k| \leqslant A_k$, for all k.

We shall prove also that the scope of Theorem C extends to a wider class than $\mathfrak{BV}$, i.e. to the class of null-sequences of bounded variation of order $m \geqslant 1$.

DEFINITION 1.4. A null-sequence $\{a_k\}$ belongs to the class $(\mathfrak{BV})^m$ if for some integer $m \geqslant 1$,

$$\sum_{k=1}^{\infty} |\Delta^m a_k| < \infty,$$

where

$$\Delta^m a_k = \Delta(\Delta^{m-1} a_k) = \Delta^{m-1} a_k - \Delta^{m-1} a_{k+1}.$$

For $m = 1$, the class $(\mathfrak{BV})^1$ is the class $\mathfrak{BV}$. It is clear that

$$\{a_k\} \in (\mathfrak{BV})^m \Rightarrow \{a_k\} \in (\mathfrak{BV})^{m+1},$$

but the converse is not true.

EXAMPLE. For $k = 1, 2, \ldots,$ and $-k \leqslant n < k$, define

$$a_{k^2+n} = (k - |n|)/k^2.$$

The sequence $\{a_i\}$ is well defined, for $k^2 + k = (k+1)^2 - (k+1)$. Since

$$|\Delta a_{k^2+n}| = 1/k^2$$

we have

$$\sum_{i=0}^{\infty} |\Delta a_i| = \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} |\Delta a_{k^2+n}|$$
$$= \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{2k}{k^2},$$

and $\{a_i\}$ is not of bounded variation. But

$$\Delta^2 a_{k^2+n} = 0,$$

for $-k < n < -1$ or $0 < n < k-2$, and

$$\Delta^2 a_{k^2-1} = \frac{-2}{k^2}, \quad \Delta^2 a_{k^2+k-1} = \frac{1}{k^2} + \frac{1}{(k+1)^2}.$$

Hence

$$\sum_{i=0}^{\infty} |\Delta^2 a_i| = \sum_{k=1}^{\infty} \sum_{n=-k}^{k-1} |\Delta^2 a_{k^2+n}|$$
$$= \sum_{k=1}^{\infty} \left[\frac{3}{k^2} + \frac{1}{(k+1)^2} \right] < 4 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

thus the sequence $\{a_i\}$ is of bounded variation of order 2, but is not of bounded variation.

Finally, we shall prove a theorem concerning the integrability of Rees-Stanojević [5] sums. An announced result of Marzuq [6] is a corollary of that theorem.

For our purpose it suffices to consider the cosine series only, for most of our results can be adapted for sine series either directly or with suitable modifications.

2. Since $\mathcal{S}$ is equivalent to $\mathcal{S}^2$, in the following theorem we prove that $\mathcal{S} \subset \mathcal{BV} \cap \mathcal{C}$ by proving that $\mathcal{S}^2 \subset \mathcal{BV} \cap \mathcal{C}$. This also provides a new proof of Theorem A.

THEOREM 1. *Let* $\{a_k\} \in \mathcal{S}^2$. *Then* $f \in L^1(0, \pi)$, *and*

$$\|s_n - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$a_n \lg n = o(1), \qquad n \to \infty.$$

PROOF. First we shall show that the point-wise limit f of

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \tag{$*$}$$

exists in $(0, \pi)$ and that f is a Fourier series, i.e. that $f \in L^1(0, \pi)$.

From $\{a_k\} \in \mathcal{S}^2$ it follows that $\{a_k\} \in \mathcal{BV}$. Indeed

$$\sum_{k=1}^{\infty} |\Delta A_k| < \sum_{k=1}^{\infty} k|\Delta A_k| < \infty.$$

On the other hand

$$nA_n < n \sum_{k=n}^{\infty} |\Delta A_k| < \sum_{k=n}^{\infty} k|\Delta A_k| = o(1).$$

Thus from

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} k\Delta A_k + nA_n$$

we get that $\Sigma_1^\infty A_k < \infty$. Since $\{a_k\} \in \mathcal{S}^2$ implies that $|\Delta a_k| < A_k$ for all k, it follows that $\{a_k\} \in \mathcal{BV}$, and hence $(*)$ converges in $(0, \pi)$ to the point-wise limit f.

It remains to show that $f \in L^1(0, \pi)$. From Theorem 2 of Garrett and Stanojević [4] we have that if $\{a_k\} \in \mathcal{BV}$, then

$$\{a_k\} \in \mathcal{C} \Rightarrow f \in L^1(0, \pi).$$

Thus, it suffices to prove that

$$\{a_k\} \in \mathcal{S}^2 \Rightarrow \{a_k\} \in \mathcal{C}.$$

Applying the Sidon-Fomin Lemma [2] we obtain

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx$$
$$< \lim_{N\to\infty} \left[\sum_{k=n+1}^{N-1} (k+1)|\Delta A_k| + (N+1)A_N + (n+1)A_{n+1} \right]$$

where C is an absolute constant. Since $NA_N = o(1)$, $N \to \infty$, we get

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \sum_{k=n+1}^{\infty} (k+1)|\Delta A_k| + (n+1)A_{n+1}.$$

Both terms on the right-hand side of the last inequality are $o(1)$, as $n \to \infty$. Thus $f \in L^1(0, \pi)$. Recalling Corollary B in [4], or Theorem B from the first section of this paper we complete the proof of Theorem 1.

To avoid technical duplications in the proof of our next two theorems, we shall prove the following proposition.

PROPOSITION. *If* $\{a_k\} \in (\mathcal{BV})^m$, *then the point-wise limit* g *of*

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} a_k + \sum_{k=1}^{n} \left(\sum_{j=k}^{n} a_k \right) \cos kx$$

exists in $(0, \pi]$.

PROOF. Let $D_k(x)$ and $\tilde{D}_k(x)$ be the Dirichlet and conjugate Dirichlet kernels, respectively. Summation by parts (with $n > m$) yields

$$g_n(x) = \sum_{k=0}^{n} a_k D_k(x).$$

If $\sum_{k=1}^{\infty}|\Delta^0 a_k| < \infty$ $(m = 0)$ we are done, since $|D_k(x)| < |1/(2 \sin x/2)|$ for $x \in (0, \pi)$. If $m > 0$ we write

$$\begin{aligned} g_n(x) &= 1/(2 \sin x/2) \sum_{k=0}^{n} a_k[\cos x/2 \sin kx + \sin x/2 \cos kx] \\ &= \frac{1}{2} \cot \frac{x}{2}\left[\sum_{k=0}^{n-1} \Delta a_k \tilde{D}_k(x) + a_n D_n(x)\right] \\ &\quad + \frac{1}{2}\left[\frac{a_0}{2} + \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x)\right]. \end{aligned}$$

If $\sum_{k=1}^{\infty}|\Delta^1 a_k| < \infty$ $(m = 1)$ we are done, since $g_n(x)$ is the sum of a constant term, partial sums of two absolutely convergent series, and nth terms of two sequences that converge to zero. If $m > 1$ we write

$$\begin{aligned} g_n(x) = \frac{1}{2} \cot \frac{x}{2}&\left\{\sum_{k=0}^{n-1} \Delta a_k[\cos x/2 - \cos(k + 1/2)x]/(2 \sin x/2) + a_n \tilde{D}_n(x)\right\} \\ &+ \frac{1}{2}\left[\frac{a_0}{2} + \sum_{k=0}^{n-1} \Delta a_k \sin(k + 1/2)x/(2 \sin x/2) + a_n D_n(x)\right] \end{aligned}$$

until we reach

$$g_n(x) = B_1(x) + A_n(x) + B_2(x) \sum_{k=0}^{n-m} \Delta_k^m D_k(x) + B_3(x) \sum_{k=0}^{n-m} \Delta^m a_k \tilde{D}_k(x)$$

where $B_1(x)$, $B_2(x)$, and $B_3(x)$ do not depend on n and $A_n(x)$ converges to zero.

In the same way, we can prove that if there exists a nonnegative integer m such that $\sum_{k=1}^{\infty}|\Delta^m a_k| < \infty$, then $s_n(x)$ converges for $x \in (0, \pi]$.

THEOREM 2. *Let* $\{a_k\} \in (\mathfrak{B}\mathfrak{V})^m$ *and* $a_n \lg n = o(1)$, $n \to \infty$. *Then* $\|s_n - f\| = o(1)$, $n \to \infty$, *if and only if* $\{a_n\} \in \mathcal{C}$.

PROOF. Using summation by parts we get

$$s_n(x) = \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x)$$

and $f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x)$ since a_n tends to zero.

For the "if" part let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon/3 \quad \text{for all } n \geqslant 0.$$

Then

$$\begin{aligned}\int_0^\pi |f(x) - s_n(x)|\,dx &= \int_0^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x) - a_n D_n(x)\right| dx \\ &< \int_0^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx + |a_n| \int_0^\pi |D_n(x)|\,dx \\ &= \int_0^\delta \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx + \int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx \\ &\quad + |a_n| \int_0^\pi |D_n(x)|\,dx \\ &< \varepsilon/3 + \int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx + \varepsilon/3\end{aligned}$$

for sufficiently large n, since $a(n) \log n$ tends to zero and

$$\int_0^\pi |D_n(x)|\,dx = o(\log n).$$

Proceeding as in the proof of the Proposition, we get

$$\begin{aligned}\int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx &< \int_\delta^\pi |A_n(x)|\,dx + \int_\delta^\pi \left|B_1(x) \sum_{k=n}^\infty \Delta^m a_k D_k(x)\right| dx \\ &\quad + \int_\delta^\pi \left|B_2(x) \sum_{k=n}^\infty \Delta^m a_k \tilde{D}_k(x)\right| dx,\end{aligned}$$

where $\int_\delta^\pi |B_1(x) D_k(x)|\,dx$ and $\int_\delta^\pi |B_2(x) \tilde{D}_k(x)|\,dx$ are both uniformly bounded and $\int_\delta^\pi |A_n(x)|\,dx$ converges to zero.

Since $\sum_{k=n}^\infty |\Delta^m a(k)| < \infty$, $\int_\delta^\pi |\sum_{k=n}^\infty \Delta a(k) D_k(x)|\,dx < \varepsilon/3$ for sufficiently large n. We have that for sufficiently large n, $\|s_n - f\| < \varepsilon$.

For the "only if" part, let $\varepsilon > 0$. Then there exists an integer N such that

$$\int_0^\pi |f(x) - s_n(x)|\,dx < \varepsilon/4 \quad \text{if } n > N.$$

That is, $\int_0^\pi |\sum_{k=n}^\infty \Delta a_k D_k(x) - a_n D_n(x)|\,dx < \varepsilon/4$ if $n > N$. Since $a_n \log n$ tends to zero and $\int_0^\pi |D_n(x)|\,dx = o(\log n)$, there exists an integer M such that $\int_0^\pi |\sum_{k=n}^\infty \Delta a_k D_k(x)|\,dx < \varepsilon/2$ if $n > M$. Now if $\sum_{k=0}^M |\Delta a_k| = 0$, then for $n < M$,

$$\int_0^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx = \int_0^\pi \left|\sum_{k=M+1}^\infty \Delta a_k D_k(x)\right| dx < \varepsilon/2 < \varepsilon.$$

If $\sum_{k=0}^M |\Delta a_k| \neq 0$, let $\delta = \varepsilon/(2M \sum_{k=0}^M |\Delta a_k|)$. For $n > M$, we have

$$\int_0^\delta \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx < \int_0^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx < \varepsilon/2 < \varepsilon.$$

For $0 \leq n < M$, we get

$$\begin{aligned}\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx &\leq \int_0^\delta \left| \sum_{k=n}^{M-1} \Delta a_k D_k(x) \right| dx + \int_0^\delta \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right| dx \\ &\leq \int_0^\delta \sum_{k=n}^{M-1} (k+1)|\Delta a_k|\, dx + \int_0^\pi \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right| dx \\ &< \delta \sum_{k=0}^{M-1} (k+1)|\Delta a_k| + \varepsilon/2 \\ &\leq \delta M \sum_{k=0}^{M-1} |\Delta a_k| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Thus the conclusion holds.

THEOREM 3. *Let $g(x)$ exist for $x \in (0, \pi]$ and $\sum_{k=0}^{\infty}(k+1)[|\Delta a_k| - t\Delta a_k] < \infty$, where $t = 1$ or $t = -1$. Then $g \in L[0, \pi]$ if and only if $\sum_{k=0}^{\infty}(k+1)[|\Delta a_k| + t\Delta a_k] < \infty$.*

PROOF. We note that sufficient conditions for $g(x)$ to exist in $(0, \pi]$ are given in the Proposition.

Let $c_n = \max\{t\Delta a_n, 0\}$ and $d_n = c_n - t\Delta a_n$. Then $2c_n = |\Delta a_n| + t\Delta a_n$ and $2d_n = |\Delta a_n| - t\Delta a_n$. Thus,

$$\begin{aligned} g(x) &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k D_k(x) \\ &= \lim_{n\to\infty} \left\{ \sum_{k=0}^{n-1} \Delta a_k (k+1) F_k(x) + a_n(n+1)F_n(x) \right\}\end{aligned}$$

where $F_n(x)$ is the Fejér kernel. Since a_n tends to zero and $(n+1)F_n(x)$ is bounded,

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} \Delta a_k (k+1) F_k(x) \\ &= \sum_{k=0}^{\infty} t[c_k - d_k](k+1)F_k(x) \\ &= \sum_{k=0}^{\infty} t c_k (k+1) F_k(x) - \sum_{k=0}^{\infty} t d_k (k+1) F_k(x).\end{aligned}$$

Now,

$$\begin{aligned} \left| \int_0^\pi \sum_{k=0}^{\infty} t d_k (k+1) F_k(x)\, dx \right| &= \left| t \sum_{k=0}^{\infty} d_k (k+1) \int_0^\pi F_k(x)\, dx \right| \\ &= \pi \sum_{k=0}^{\infty} d_k (k+1) < \infty.\end{aligned}$$

Thus, $g \in L^1[0, \pi]$ if and only if

$$\left| \int_0^\pi \sum_{k=0}^{\infty} t c_k (k+1) F_k(x)\, dx \right| = \pi \sum_{k=0}^{\infty} c_k (k+1) < \infty.$$

Theorem 3 also generalizes the following result of Marzuq [6]: Let $\{a_k\}$ be a nonnegative quasi-monotone sequence tending to zero and let $\sum_{k=1}^{\infty} a_k/k < \infty$ and $\sum_{k=1}^{\infty}(k+1)[|\Delta a_k| - \Delta a_k] < \infty$. Then $g \in L^1[-\pi, \pi]$ if and only if $\sum_{n=1}^{\infty} a_n < \infty$.

Note that under the Marzuq condition we have $\sum_{k=1}^{\infty}|\Delta a_k| < \infty$, for $|\Delta a_k| = |\Delta a_k| - \Delta a_k + \Delta a_k$. Now $|\Delta a_k| - \Delta a_k$ is 0 if $\Delta a_k \geqslant 0$ and is $-2\Delta a_k \leqslant 2\alpha a_k/k$ if $\Delta a_k < 0$. Thus, $\sum_{k=1}^{\infty}|\Delta a_k| \leqslant 2\alpha\sum_{k=1}^{\infty} a_k/k + \sum_{k=1}^{\infty}\Delta a_k < \infty$.

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